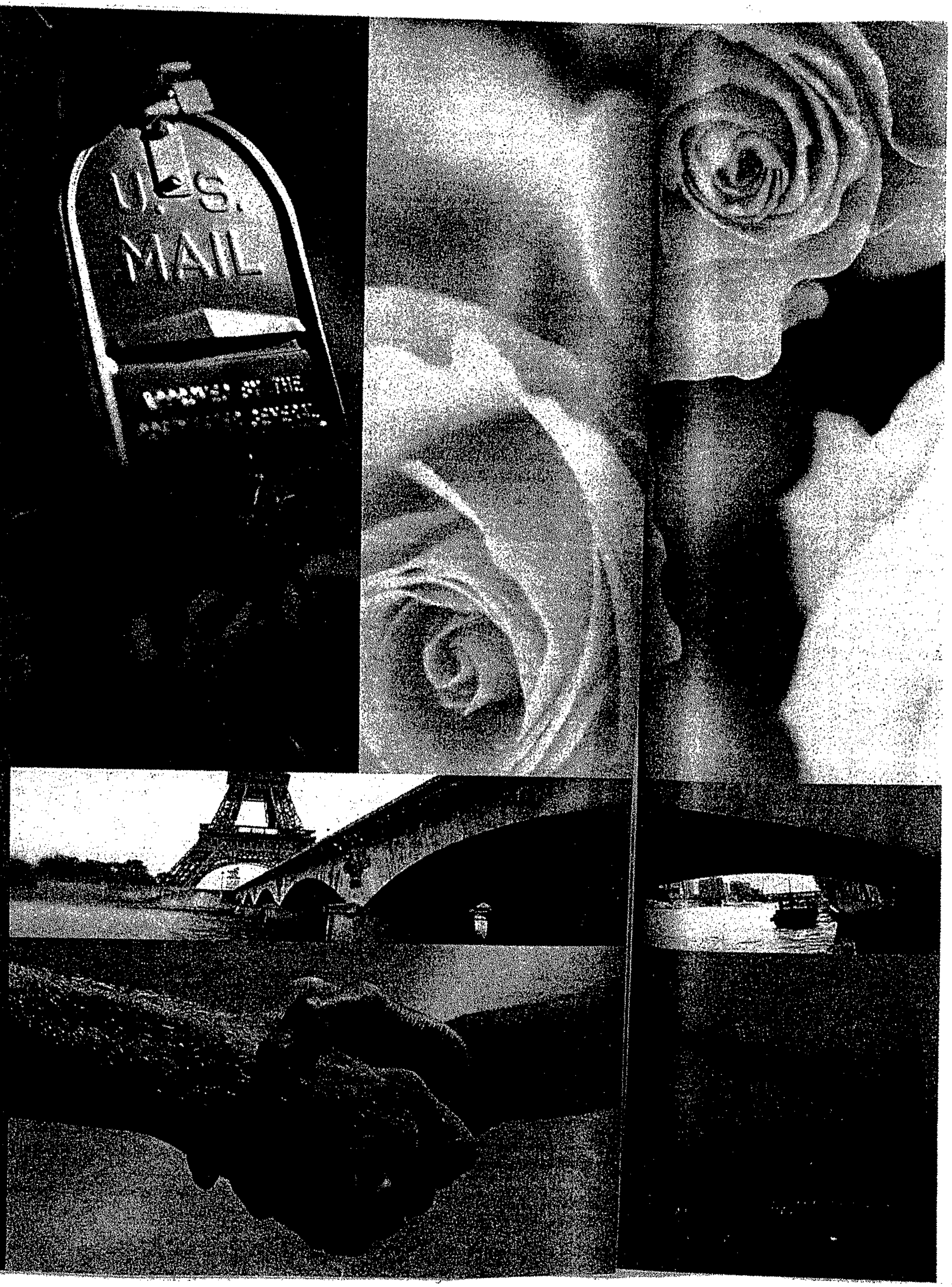
CHAPTER 11



GRAPHS AND CIRCUITS

The city of Kaliningrad in Russia is situated where two branches of the Pregol'a River come together. In 1736, this city was called Königsberg and was a part of East Prussia ruled from what is Germany today. At that time, parts of Königsberg were on the banks of the river, another part was on a large island in the middle, and a final part was between the two branches of the river. Seven bridges connected these four parts of the city. An unsolved problem of the time was:

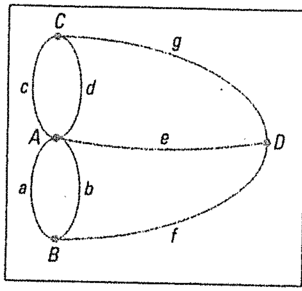
Is it possible for a person to walk around the city traversing each bridge exactly once, starting and ending at the same point?

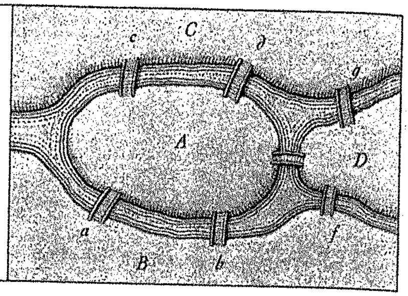
This problem is now known as the **Königsberg bridge problem**. It became famous because it was the subject of a research paper in that year by the great mathematician Leonhard Euler. A drawing like the one on page 658 was included in that paper. Take a minute or two to see if you can find such a walk.

To solve the Königsberg bridge problem, Euler constructed a simple and helpful geometric model of the situation called a *graph*, and his paper is usually acknowledged to be the origin of the subject called **graph theory**. (As you will see, these graphs are not the same as the graphs of functions or relations.) In the two-and-a-half centuries since his solution, graphs have been used to solve a wide variety of problems. In this chapter, you will be introduced to a selection of those problems.

LESSON

11-1 Modeling with Graphs





Traversing the Edges of a Graph

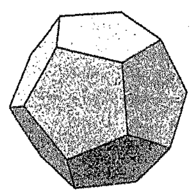
To solve the Königsberg bridge problem of the preceding page, Euler observed that, for this problem, each land mass could be represented by a point since it is possible to walk from any part of a land mass to any other part without crossing a bridge. The bridges could be thought of as arcs joining these points. Thus the situation of Königsberg could be represented by the following geometric model consisting of four points and seven arcs.

This type of geometric model is called a graph. The four points are the **vertices** and the seven arcs are the **edges** of the graph. In terms of this graph, the Königsberg bridge problem can be stated:

> Is it possible to trace this graph with a pencil, traveling each edge exactly once, starting and ending at the same vertex, without picking up the pencil?

Euler's solution to the Königsberg bridge problem is given in Lesson 11-4.

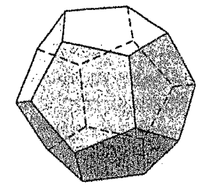
Traversing the Vertices of a Graph

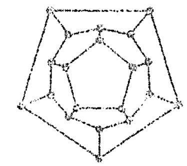


Another famous puzzle for which a graph is a very helpful model was invented in 1859 by the Irish mathematician Sir William Rowan Hamilton (1805–1865). The puzzle consisted of a wooden block in the shape of a regular dodecahedron, as shown at the right.

This polyhedron has 12 regular pentagons as its faces, 30 edges, and 20 vertices. Hamilton marked each vertex of the block with the name of a city, and the object of the puzzle was to find a travel route along the edges of the block that would visit each of the cities once and only once. A small pin protruded from each vertex so that the player could mark a route by wrapping string around each pin in order as its city was visited.

By thinking of the polyhedron as transparent, as shown at the top of page 659 at the left, you can count to determine that Hamilton's problem involves a graph with 20 vertices and 30 edges in which 3 edges meet at each vertex. The graph at the right shows the same relationships of vertices and edges in a 2-dimensional diagram; the two graphs are **equivalent**.





Hamilton's puzzle can be stated in terms of either graph as the following problem:

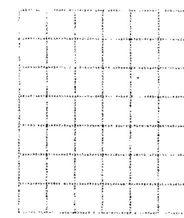
> Is it possible to trace this graph with a pencil, traveling through each *vertex* exactly once, without picking up the pencil?

In fact, Hamilton also sold the 2-dimensional version of his puzzle since it was easier to work with. Notice the similarity between the problems of Euler and Hamilton; in both the goal is to traverse all objects of one kind in the graph exactly once; in Euler's problem the objects are the edges; in Hamilton's the objects are the vertices.

Several practical situations are examples of Hamilton's problem.

Example 1

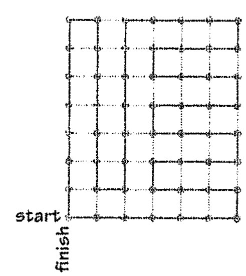
One mailbox is located at each intersection of a city and a postal truck is required to collect the mail from all the boxes in the 42-block region in the map at the right.



Can the driver plan a pick-up route that begins and ends at the same place and allows collection of the mail from all of the boxes without passing one that has already been collected?

Solution

The problem can be modeled by a graph, with the mailboxes being the vertices and the streets being the edges. Experimentation gives several suitable routes such as the one pictured here.

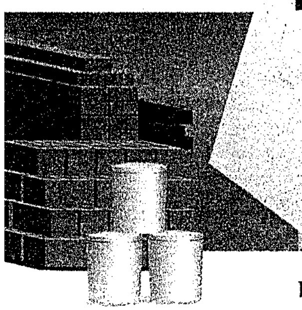


A Non-Geometric Example

The two examples you have seen so far result from situations that themselves are geometric. The next example is quite different; it shows the use of graphs in scheduling a complex task. First, some background information is necessary.

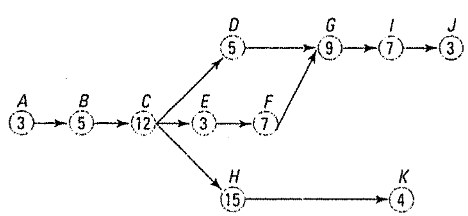
Building a house is usually a team effort that involves specialists such as architects, excavators, concrete workers, framing carpenters, drywallers, electricians, plumbers, roofers, heating and air conditioning workers, finish carpenters, painters, and landscapers. Different specialists are often able to work at the same time provided that the work that must precede a particular specialist is completed before that specialist begins. By working simultaneously whenever possible, the house can be completed more quickly, and it is natural to wonder if there is some optimal way to schedule the various tasks for completion.

This can be done with a graph, but first the information to be graphed must be assembled. Here is a list of some of the tasks involved in building a house, along with the time they require and the tasks which must be completed prior to their beginning. (The list is simplified but the ideas are not.)



Task		Time (days)	Prerequisite tasks
A	Preparing final house and site plans	3	none
B	Excavation and foundation construction	5	*A*
C	Framing and closing main structure	12	*B*
D	Plumbing	5	*C*
E	Wiring	3	*C*
F	Heating-cooling installation	7	*E*
G	Insulation and dry wall	9	*D, F*
H	Exterior siding, trim, and painting	15	*C*
I	Interior finishing and painting	7	*G*
J	Carpeting	3	*I*
K	Landscaping	4	*H*

It would take 73 days to finish the house if only one task were done on any day. However, the following graph can help the builder decide which tasks can be done simultaneously in order to complete the job more quickly. The tasks are represented by vertices, drawn here as circles. The number of days needed for each job is indicated inside its circle. The arrows represent edges; when an arrow is drawn such as from vertex *A* to vertex *B*, it means that task *A* must be completed before task *B* can begin.

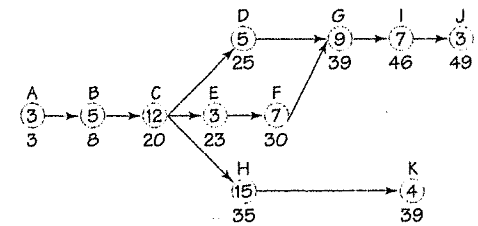


Example 2

Use the graph on page 660 to determine the least number of days to complete the house.

Solution

From left to right along the graph, calculate for each task the least number of days in which it can be completed since the beginning of construction. Write this number beneath the circle representing the task. For instance, task *D* requires the completion of tasks *A, B,* and *C,* which takes 20 days. Since *D* requires 5 more days, write 25 beneath *D*. Through Task *F* similarly requires 30 days. Since both *D* and *F* must be done before *G, G* cannot be started before 30 days and will not be completed before 39 days.



All the tasks will be completed when both *J* and *K* are done. Since *J* requires at least 49 days and *K* requires at least 39 days for its completion, ***the house can be completed in 49 days.***

Notice that the algorithm used in Example 2 is recursive. Here is the general algorithm to calculate the number of days for a particular task:

> If there are no prerequisite tasks, use the number of days required by this task alone.
>
> Otherwise:
>
> (1) Calculate the number of days for each prerequisite task by using this algorithm.
>
> (2) Choose the largest of the numbers found in step (1), and add to it the number of days required by this task alone.

This algorithm is used to determine efficient job schedules for much more complex projects, and there exists computer software which will automatically create the graph and find the solution after the user inputs information like that given in Example 2.

The graph of Example 2 is different from the others in this lesson in that you can travel along each edge in only one direction. Such graphs are called **directed graphs** or **digraphs**. The graph in the solution differs in a second way; each vertex is labeled with a number.

A Probability Tree

A particular type of digraph, called a *probability tree*, is useful for solving certain problems involving probabilities. In a **probability tree**, a vertex represents an event, and the edge leading from vertex A to vertex B is labeled with the probability that event B occurs if A occurs. The last vertex of each branch is called a *leaf* of the tree.

Example 3

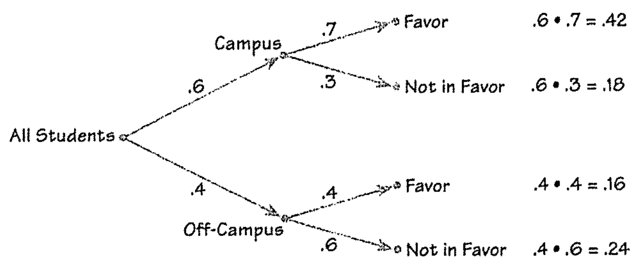
60% of the students in a college live on campus. 70% of those who live on campus favor improved student health services even if it means an increase in tuition. 40% of those who live off campus favor this increase. Draw a graph to answer the following.

a. What is the proportion of students who favor this increase to pay for improved student health services?

b. If a randomly selected student favors the tuition increase, what is the probability that the student lives on campus?

Solution

a. Draw a graph showing the division of students by residence: Campus or Off-Campus. Since 60% of the students live on campus, 40% must live off campus. Then break down each group by their opinions on the increase. The proportion of students who are both on campus and favor the increase is $.6 \cdot .7 = .42$. The proportion of students who are on campus and don't favor the increase is $.6 \cdot .3 = .18$. Similarly, you can compute the proportion of students in the two branches leading from the Off-Campus vertex.



Notice that the proportions at the end of the graph add to 1.00; all students are represented by the leaves of the tree. **The proportion of students who favor the tuition increase is .42 + .16 or 58% of the student body.**

b. 58% of the student body are in favor of the increase, while 42% are in favor *and* live on campus. Therefore, **given that a student favors the increase, there is a $\frac{.42}{.58} \approx 72\%$ probability that the student lives on campus.**

QUESTIONS

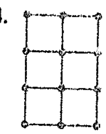
Covering the Reading

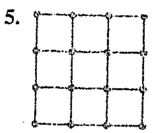
1. The seventh bridge of Königsberg to be built connected the land masses B and D in the drawing on page 658.
   a. Draw a graph of the Königsberg bridge problem when the city had only 6 bridges.
   b. Show that there is a path around the city crossing each of the six bridges exactly once by listing the land masses and bridges, in order. Does your path begin and end at the same point?

2. Copy the graph for Hamilton's dodecahedron. Determine whether there is a solution to Hamilton's problem.

3. In Euler's problem, each __?__ is traversed exactly once, and in Hamilton's problem, each __?__ is traversed exactly once.

In 4 and 5, copy the graph. Think of the vertices of the graph as mailboxes and the edges as streets, and draw a path that will allow a driver to collect the mail from each box without passing one whose mail has already been collected, starting and finishing at the same point.

4. 

5. 

6. In the situation of Example 2, a new task L, the moving in of appliances, taking one day, and requiring the completion of task I, is added to the schedule.
   a. Draw a new graph including task L.
   b. How does this affect the time to finish the house?

In 7–9, refer to Example 3.

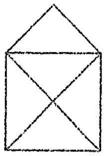
7. If there are 2000 students at the college, how many live off campus *and* favor improved student health services?

8. What is the proportion of students who don't favor a tuition increase for improved student health services?

9. If a student selected at random doesn't favor a tuition increase for improved student health services, what is the probability that he or she lives on campus?

10. A store estimates that 80% of its customers are female. The probability that a female wears contact lenses is 6%. The probability that a male wears contact lenses is 5%.
   a. Draw a probability tree and label its edges with the appropriate probabilities to represent this situation.
   b. What proportion of customers of this store are male and don't wear contact lenses?
   c. What proportion wear contacts?

Applying the Mathematics



11. A common children's puzzle is to traverse the edges of the drawing at the left exactly once. Is this puzzle more like Euler's problem or Hamilton's problem?

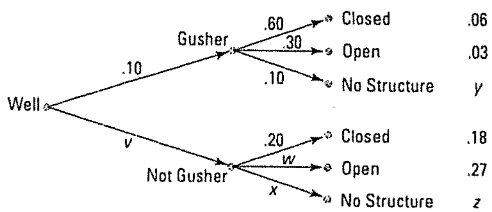
12. The possibility tree of Example 1 in Lesson 10-2 is a graph.
    a. How many vertices does that graph have?
    b. How many edges does that graph have?

Oil Rig. *This roughneck, drenched in mud which acts as a lubricant, is attaching a new section of pipe.*

13. An oil company has used expensive test-drilling to determine what areas of its land will be worth developing as oil fields. Only 10% of the test wells are gushers which indicate oil deposits worth developing. Test wells are extremely expensive to drill, so the company has been studying the use of less expensive seismic tests on a plot to predict whether drilling is worthwhile. Seismic tests indicate whether the underlying strata have a closed structure, open structure, or no structure. Over a period of time, the company performs seismic tests in conjunction with drilling. The table below shows the proportions of gushers and of nongushers that are associated with each structure. For example, 60% of the gushers came from closed structures.

	Closed	Open	No Structure
Gusher	.60	.30	.10
Not a gusher	.20	.30	.50

a. Find the probabilities v, w, x, y, and z in the following probability tree.



b. What is the probability that the underlying strata of a randomly selected well has a closed structure?
c. Suppose the company conducts a seismic test on one of its properties and finds a closed structure. According to the data above, what is the probability that drilling there would produce a gusher?
d. Suppose a seismic test indicates an open structure. What is the probability that drilling there would produce a gusher?

14. In Queenstown there is a river with two islands and bridges connecting the islands to the shore, as depicted at the right. Is there a path traversing each bridge exactly once?

15. Draw a digraph whose vertices represent the following nine types of figures: isosceles trapezoids, kites, parallelograms, polygons, quadrilaterals, rectangles, rhombuses, squares, and trapezoids. Draw an arrow connecting vertex X to vertex Y if and only if all Xs are Ys, but do not connect X to Z if X is connected to Y and Y is connected to Z.

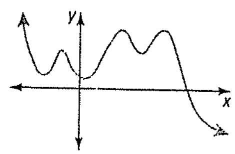
Review

16. a. Sketch the graph of a function whose first derivative is negative on the interval [-5, -2) and positive on the interval (-2, 3].
    b. Identify the location of a relative minimum or maximum. Which is it? *(Lesson 9-5)*

17. Consider the matrix at the right.
    a. What are its dimensions?
    b. If a_{ij} is the element in the ith row and jth column, what is a_{23}?
    c. Calculate $\sum_{i=1}^{3} a_{ii}$. *(Lesson 7-2)*

$$\begin{bmatrix} 5 & 1 & 0 \\ 3 & 2 & 0 \\ 8 & 0.5 & -3 \\ 4 & 6 & 10 \end{bmatrix}$$

18. Express in terms of $\tan x$. *(Lesson 6-5)*
    a. $\tan\left(x + \frac{\pi}{2}\right)$ b. $\tan(x + \pi)$ c. $\tan\left(x + \frac{3\pi}{2}\right)$



19. Suppose p is a polynomial function whose graph is shown at the left. What is the smallest possible value for the degree of p? *(Lesson 4-4)*

20. Solve $\sqrt{3y + 1} = 4$. Indicate which steps are reversible and which are not. *(Lesson 3-3)*

Exploration

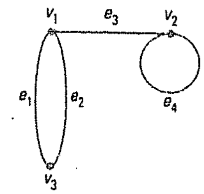
21. Example 1 and Questions 4 and 5 ask for the traversing of all the vertices of a rectangular array exactly once, going horizontally or vertically, and beginning and ending at the same point. In size, these arrays are 8 by 7, 4 by 3, and 4 by 4.
    a. Find a rectangular array that cannot be traversed in this way.
    b. Find a criterion that seems to distinguish the dimensions of those arrays that can be traversed from those which cannot be traversed.

LESSON

11-2 The Definition of Graph

What Exactly Is a Graph?

In the last lesson you saw examples of problems that could be solved using graphs, but no definition of *graph* was given. In order to deduce general properties of graphs, a definition is needed. Consider the graph below.



(1) Its vertices are v_1, v_2, and v_3.
(2) Its edges are e_1, e_2, e_3, and e_4.
(3) The endpoints of each edge are given by the following table.

edge	endpoints
e_1	$\{v_1, v_3\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_1, v_2\}$
e_4	$\{v_2\}$

This table describes an *edge-endpoint function.*

To specify a graph it is necessary to provide the kind of information given in (1)–(3) above, with words or a picture. Here is a formal definition of *graph.*

Definition
A **graph** G consists of
1. a finite set of vertices,
2. a finite set of edges,
3. a function (the **edge-endpoint function**) that maps each edge to a set of either one or two vertices (the **endpoints** of the edge).

Some Vocabulary of Graphs

In a graph there must be a vertex at each end of every edge. An edge *connects* or *joins* its endpoints, but has no points other than its endpoints. So its shape—curve or segment—is not important. Two vertices connected by an edge are **adjacent vertices**. Two edges with a common endpoint are called **adjacent edges**. For instance, e_3 and e_2 above are adjacent edges.

Note that vertices v_1 and v_3 are connected by more than one edge. When this occurs the edges are said to be **parallel**. (This is a different meaning for "parallel" than that associated with lines.) Also edge e_4 joins vertex v_2 to itself. Such an edge is called a **loop**.

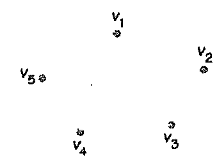
Example 1

Draw a picture of the graph G defined as follows.
1. set of vertices: $\{v_1, v_2, v_3, v_4, v_5\}$
2. set of edges: $\{e_1, e_2, e_3, e_4, e_5\}$
3. edge-endpoint function:

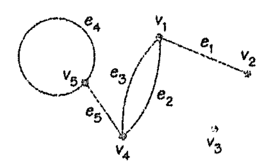
edge	endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_4\}$
e_3	$\{v_1, v_4\}$
e_4	$\{v_5\}$
e_5	$\{v_4, v_5\}$

Solution

It is often convenient to start the graph by placing the vertices as though they were consecutive vertices of a convex polygon. Since there are 5 vertices, begin with a pentagon.



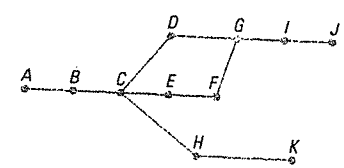
Then fill in edges as specified by the edge-endpoint function table.



In Example 1, vertex v_3 is not the endpoint of any edge. It is called **isolated**. The definition of graph allows isolated vertices. Although all edges must have endpoints, a vertex need not be the endpoint of an edge.

Simple Graphs

A picture of the graph from the housebuilding problem in Lesson 11-1 is repeated here with the vertices shown by dots.



In this graph, there are no loops or parallel edges. Such a graph is called *simple.*

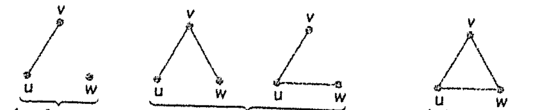
Definition
A graph is **simple** if and only if it does not have loops and it does not have parallel edges.

In a simple graph with vertices v and w, if edge $\{v, w\}$ exists, it is unique, since there is at most one edge joining any two of the graph's vertices.

Example 2

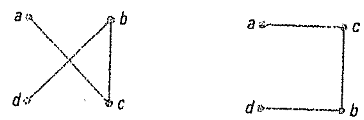
Draw all simple graphs with vertices $\{u, v, w\}$ if one of the edges is $\{u, v\}$.

Solution



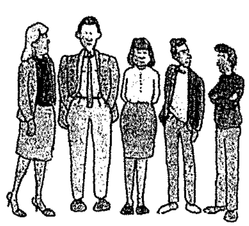
$\{u, v\}$ is the only edge. — There is one other edge besides $\{u, v\}$. — There are two other edges besides $\{u, v\}$.

Consider the simple graph with vertices a, b, c, d and edges $\{a, c\}$, $\{b, d\}$, and $\{b, c\}$. Two pictures of this graph are shown below. In the left picture, the pictures of edges $\{a, c\}$ and $\{b, d\}$ intersect but the edges do not (because they have no points other than their endpoints). Such intersections in pictures are called **crossings**. The figure at the right illustrates the same graph but it avoids crossings.

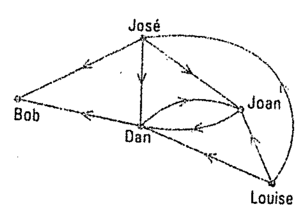


Digraphs

Recall from Lesson 11-1 that it is sometimes useful to add direction to each edge of a graph. The resulting digraph is pictured like other graphs except that its edges are drawn with arrows. The formal definition of *digraph* is the same as the definition of graph except that the edge-endpoint function sends each edge to an *ordered pair* of vertices.



For instance, some group-behavior studies investigate the influence one person has on another in a social setting. The directed graph pictured at the right shows such a set of influence relationships.



The arrows indicate that, for example, José influences Bob, Dan, and Joan, and Louise influences José, Dan, and Joan. The edge connecting Louise and Joan is said to go *from* the vertex for Louise *to* the vertex for Joan.

Using a Matrix to Describe a Graph

It is natural to want to describe a graph numerically. It may be somewhat surprising that this can be done using a matrix.

Definition
The **adjacency matrix** M for a graph with vertices $v_1, v_2, \ldots, v_n$ is the $n \times n$ matrix in which, $\forall\, i$ and j, the element in the ith row and jth column is the number of edges from vertex v_i to vertex v_j.

Example 3

Write the adjacency matrix for the directed graph of influence relationships pictured on page 668.

Solution

Because there are 5 vertices, the adjacency matrix has 5 rows and 5 columns. Label the rows and columns with the vertex names. To fill in the entry in the ith row and jth column, just count the number of edges from v_i to v_j. For instance, there are no edges from v_1 to v_1, so the entry in row 1, column 1 is 0. There is one edge from v_2 to v_3, so the entry in row 2, column 3 is 1. The entire adjacency matrix is given below.

$$\begin{array}{r c}
 & \begin{array}{ccccc} \text{Bob} & \text{José} & \text{Dan} & \text{Joan} & \text{Louise} \\ v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\
\begin{array}{r} \text{Bob} = v_1 \\ \text{José} = v_2 \\ \text{Dan} = v_3 \\ \text{Joan} = v_4 \\ \text{Louise} = v_5 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}
\end{array}$$

Check

In the matrix for a directed graph, each edge appears once, so the sum of all elements of the matrix equals the number of edges of the graph. Here the sum in the matrix is 9, which checks with the drawing on page 668.

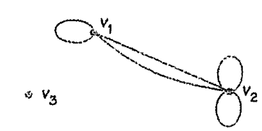
$$\begin{array}{r c}
 & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\
\begin{array}{r} v_1 \\ v_2 \\ v_3 \end{array} & \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{array}$$

Example 4

Draw a picture of a graph (*not* directed) that has the adjacency matrix shown at the left.

Solution

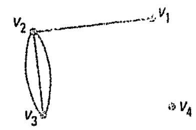
Draw vertices v_1, v_2, and v_3 and connect them by edges as indicated in the matrix. For example, the 2 in the first row and second column indicates that two edges should go from v_1 to v_2. Since the graph is not directed, these two edges also go from v_2 to v_1, agreeing with the 2 in the second row and first column of the matrix. Note that when the graph is completed there is one loop at v_1 and two at v_2, that there are parallel edges joining v_1 and v_2, and that v_3 is an isolated vertex.



QUESTIONS

Covering the Reading

1. Refer to the figure at the right.
   a. Does the figure represent a graph?
   b. If it does, tell how many edges and how many vertices it has. If it does not, explain why not.



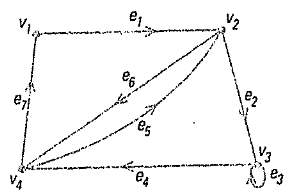
2. Draw a picture of the graph G defined as follows.
   (1) set of vertices: $\{v_1, v_2, v_3, v_4, v_5\}$
   (2) set of edges: $\{e_1, e_2, e_3, e_4, e_5, e_6\}$
   (3) edge-endpoint function: (at right)

edge	endpoints
e_1	$\{v_1, v_3\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_2\}$
e_4	$\{v_2\}$
e_5	$\{v_3, v_5\}$
e_6	$\{v_3, v_5\}$

3. Use the graph G of Question 2.
   a. Are edges e_1 and e_5 adjacent?
   b. Are vertices v_1 and v_2 adjacent?
   c. Identify all isolated vertices.
   d. Identify all parallel edges.
   e. Identify all loops.

4. *True or false.* The directed graph following Example 2 shows that Bob influences Dan.

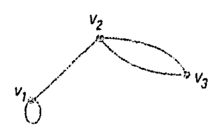
5. Write the adjacency matrix for the directed graph pictured at the right.



6. Draw a directed graph with the adjacency matrix shown below.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

7. Write the adjacency matrix for the graph pictured at the right.



8. Give the numbers of vertices and edges of the graph in Example 2 of Lesson 11-1.

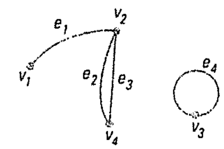
9. Draw all simple graphs with vertices $\{a, b, c, d\}$, one edge $\{a, b\}$, and two other edges.

Applying the Mathematics

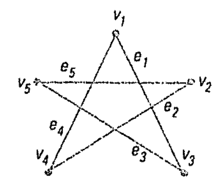
10. a. Does the adjacency matrix below represent a simple graph?
    b. Explain your answer.

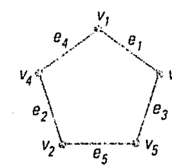
$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

11. Construct an edge-endpoint function table for the graph pictured at the right.



12. Do the following two pictures represent the same graph? Explain your answer.



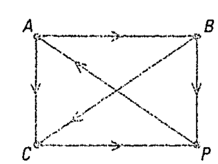


13. a. Write the negation of the following statement:
    $\forall$ *graphs G, if G does not have any loops, then G is simple.*
    b. Is the statement you obtained in part **a** *true* or *false*? Justify your answer.

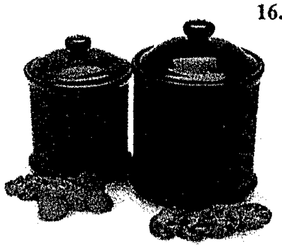
14. The edge-endpoint function for a directed graph sends each edge to an ordered pair of vertices. For instance, suppose edge e is sent to the *ordered pair* $\{u, v\}$. When a picture of this graph is drawn, there will be an arrow pointing from u to v to show that the edge e goes from u to v. Draw a picture of the directed graph defined as follows.
    vertices: $\{v_1, v_2, v_3, v_4\}$
    edges: $\{e_1, e_2, e_3, e_4, e_5, e_6\}$
    edge-endpoint function:

edge	endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_3, v_1\}$
e_4	$\{v_3, v_3\}$
e_5	$\{v_2, v_4\}$
e_6	$\{v_3, v_4\}$

15. A food P is being test-marketed against 3 leading brands A, B, and C. At the right, vertices A, B, C, and P represent the products, and an arrow is drawn from vertex x to vertex y if the taster prefers x to y. Explain why there is an inconsistency in the taster's preferences.

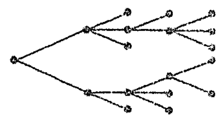


Review



16. The green cookie jar in the kitchen contains five chocolate chip cookies and seven peanut butter cookies. The red cookie jar contains three chocolate chip cookies and five vanilla wafers. In the middle of the night little Freddy sneaks into the kitchen. He doesn't turn on the light for fear of waking his parents. He puts his hand in one of the jars at random and pulls out a cookie. *(Lesson 11-1)*
   a. Draw a probability tree and label its edges with probabilities to represent this situation.
   b. What is the probability that he gets a chocolate chip cookie?
   c. When he bites into the cookie, he finds it is chocolate chip. What is the probability that it came from the red jar?

17. a. How many vertices does the possibility tree shown at the right have?
   b. How many edges does it have?
   c. Make a conjecture about the number of vertices and edges in a possibility tree, based on parts **a** and **b**, and on Question 12 of Lesson 11-1. *(Lesson 11-1)*



18. Let $p(x)$ be a polynomial with real coefficients such that $p(2 - i) = 0$ and $p(-3) = 0$. Find a possible formula for $p(x)$. *(Lesson 8-9)*

19. Find the quotient and remainder when $p(x) = 6x^4 - 7x^2 + 3x + 1$ is divided by $d(x) = x + .7$. *(Lesson 4-3)*

20. a. Multiply the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ by itself and simplify the result.
   b. Generalize the result in part **a**. *(Previous course)*

Exploration

21. a. Consider simple graphs that have four vertices $\{a, b, c, d\}$, and at least the edges $\{a, b\}$ and $\{b, c\}$. How many such graphs have exactly the indicated number of edges?
   i. 2 ii. 3 iii. 4 iv. 5 v. 6
   b. Consider simple graphs which have vertices $\{a, b, c, d\}$ and at least the edges $\{a, b\}$, $\{b, c\}$, and $\{c, d\}$. How many such graphs have exactly the indicated number of edges?
   i. 3 ii. 4 iii. 5 iv. 6
   c. What do the sequences of answers to parts **a** and **b** suggest?
   d. How is Example 2 and its answer similar to this problem?

LESSON

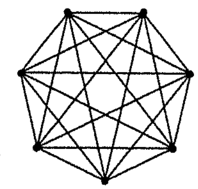
11-3 *Handshake Problems*

Bill Clinton shaking hands with the crowd at the Lyndon Baines Johnson Library in Austin, TX, while he was running for president.

The Classic Handshake Problem

Suppose n people are at a party. If each person shakes hands with every other person, how many handshakes are required?

This *handshake problem* can be represented by a graph in the following way. Represent the n people by vertices and join two vertices by an edge if the corresponding people shake hands. Since each person shakes hands once with each other person, every pair of vertices is joined by exactly one edge. A graph with this property is called a **complete graph**. Here is a picture of the complete graph with 7 vertices.



You can see that a complete graph can be pictured as the union of a polygon with its diagonals. Thus the number of edges in the complete graph with n vertices equals the total number of sides and diagonals in an n-gon, which in turn is the answer to the handshake problem.

There are a number of ways to solve the handshake problem. One way is to use combinations. Note that there are as many handshakes (edges) as there are ways to choose 2 people (vertices) to shake hands out of a group of n. This number is $\binom{n}{2}$, which equals $\frac{n(n-1)}{2}$. For the above graph, when $n = 7$, there are 21 handshakes, corresponding to the total of 7 sides and 14 diagonals for a heptagon.

Many other problems are equivalent to handshake problems. For instance, replacing handshakes by games and people by teams converts any handshake problem into a problem involving games and teams. The solution to the above problem implies that $\frac{n(n-1)}{2}$ games are required for each of n teams to play each of the other teams exactly once.

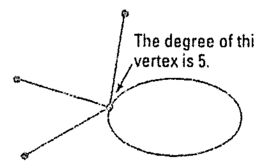
The Degree of a Graph

Here is a different handshake problem, one which seems more difficult.

> Forty-seven people attend a social gathering. During the course of the event, various people shake hands. Is it possible for each person to shake hands with exactly nine other people?

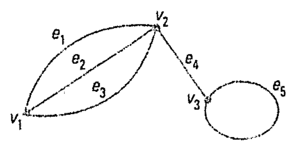
The concept of *degree*, together with properties of even and odd integers, can help to solve this problem.

Definition
If v is a vertex of a graph G, the **degree of v**, denoted **deg(v)**, equals the number of edges that have v as an endpoint, with each edge that is a loop counted twice.



Consider the graph G pictured at the right. Its vertices have the following degrees:

$$\deg(v_1) = 3$$
$$\deg(v_2) = 4$$
$$\deg(v_3) = 3.$$



The Total Degree of a Graph

The **total degree of a graph** is the sum of the degrees of all the vertices of the graph. Thus the total degree of graph G above is 10, which is twice the number of edges. Is this always the case? The answer is yes. The reason is that each edge of a graph contributes 2 to the total degree whether or not the edge is a loop. For instance, in the graph pictured above,

Edge	contributes 1 to the degree of	and 1 to the degree of
e_1	v_1	v_2
e_2	v_1	v_2
e_3	v_1	v_2
e_4	v_2	v_3
e_5, a loop, contributes 2 to the degree of v_3.		

This argument proves the following theorem.

Theorem (Total Degree of a Graph)
The total degree of any graph equals twice the number of edges in the graph.

The theorem has corollaries, which you are asked to prove in the Questions.

Corollaries
1. Total Degree Is Even: The total degree of any graph is an even positive integer.
2. Number of Odd Vertices Is Even: Every graph has an even number of vertices of odd degree.

The second corollary helps to answer the question about handshakes at a social gathering.

Example 1

In a group of 47 people, can each person shake hands with exactly nine other people? Explain why or why not.

Solution

The answer is *no*. To see why, *assume that each of the 47 people could shake hands with exactly 9 others. Represent each person as the vertex of a graph, and draw an edge joining each pair of people who shake hands. The graph would then have 47 vertices, each of which would have degree 9. So it would have an odd number of vertices of odd degree. But this contradicts the second corollary, so the assumption must be false. Thus it is impossible for each of 47 people to shake hands with exactly 9 others.*

Drawing Graphs with Vertices of Specified Degrees

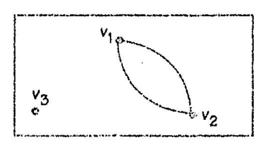
Example 2

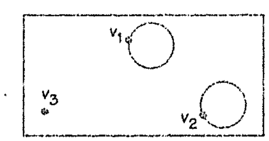
In parts **a** and **b**, draw the specified graph, or show that no such graph exists.

a. a graph with three vertices of degrees 2, 2, and 0
b. a simple graph with three vertices of degrees 2, 2, and 0

Solution

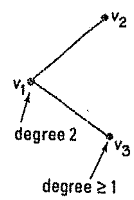
a. This combination of degrees is not forbidden by the Total Degree of a Graph Theorem because the total degree of the graph would be 4, which is even. The number of edges in the graph would be half the total degree, or 2. If you experiment by drawing three vertices connected in various ways by two edges, you quickly find that each of the graphs below satisfies the given properties.





b. Neither graph in part **a** is simple. If you continue to experiment by shifting the positions of the edges in the graphs above, you continually come up with graphs that either are not simple or have vertices with different degrees than are required. At a certain point, you would probably conjecture that no such graph exists. Proof by contradiction is a natural approach to use to prove this conjecture.

To prove:
There is no simple graph with three vertices of degrees 2, 2, and 0.



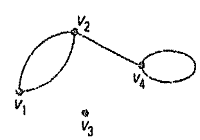
Proof (by contradiction): *Assume that there is a simple graph with three vertices of degrees 2, 2, and 0.* (A contradiction must be deduced.) *Let G be such a graph and let its vertices of degree 2 be v_1 and v_2 and its vertex of degree 0 be v_3. Since v_1 has degree 2 and G has no loops or parallel edges (because it is simple), there must be edges joining v_1 to v_2 and v_1 to v_3. Consequently, the degree of v_3 will be at least 1. On the other hand, the degree of v_3 is required to be 0. This is a contradiction, so the assumption that there is such a simple graph is false and the conjecture that no such graph exists is true.*

The result of Example 2b can be put into the language of handshakes. In a group of 3 people in which no pair shakes hands twice, it is impossible for 2 people to shake hands with 2 others and the third person not to shake any hands at all.

QUESTIONS

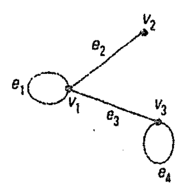
Covering the Reading

1. Consider the graph G pictured at the right.
   a. Find the degree of each vertex.
   b. What is the total degree of G?



2. Explain why the following statement is *false*: The degree of a vertex equals the number of edges that have the vertex as an endpoint.

3. Consider the graph pictured at the right. Fill in the table below for this graph.



Edge e_1 contributes __a.__ to the degree of __b.__
Edge e_2 contributes __c.__ to the degree of __d.__ and __e.__ to the degree of __f.__
Edge e_3 ____________g.____________
Edge e_4 ____________h.____________

What's Buzz'n, cousin?
At their annual reunion, members of the Limon family look over the family tree.

4. How many handshakes are needed for 23 people at a party if each person is to shake hands with every other person?

5. Draw a complete graph with 5 vertices.

6. At a family reunion, 8 cousins wish to reminisce with each other two at a time.
   a. How many conversations are needed?
   b. Verify your answer to part **a** with a graph.
   c. Explain how this problem is equivalent to a handshake problem.

7. What is the total degree of the graph given at the beginning of this lesson?

8. What is the total degree of the second graph of this lesson?

9. Prove that, in a group of 9 people, it is impossible for every person to shake hands with exactly 3 others.

10. Correct this false statement. *The number of edges in any graph equals twice the total degree of the graph.*

Applying the Mathematics

11. Prove or disprove. *A graph must have an odd number of vertices of even degree.*

In 12–15, either draw a graph with the specified properties or explain why no such graph exists.

12. a graph with 10 vertices of degrees 1, 1, 1, 2, 3, 3, 3, 4, 5, 6

13. a graph with 4 vertices of degrees 1, 1, 3, and 3

14. a simple graph with 4 vertices of degrees 1, 1, 2, and 2

15. a simple graph with 4 vertices of degrees 1, 1, 3, and 3

16. Let G be a simple graph with n vertices.
    a. What is the maximum degree of any vertex of G?
    b. What is the maximum total degree of G?
    c. What is the maximum number of edges of G?

17. Use the answer to the handshake problem at the beginning of the lesson to deduce an expression for the number of diagonals of an n-gon.

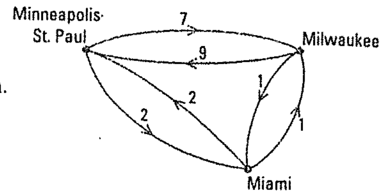
18. At a party, the first guest to arrive shakes hands with the host. The second guest shakes hands with the host and the first guest, and so on.
    a. If there are n guests, how many handshakes are there in all?
    b. Relate part **a** to the first handshake problem of this lesson.

19. Explain why the Total Degree Is Even corollary follows immediately from the Total Degree of a Graph Theorem.

20. Explain how the Number of Odd Vertices Is Even corollary follows from the Total Degree of a Graph Theorem.

Review

21. The numbers by the arcs show the number of daily non-stop flights between those cities when this question was written. Write the adjacency matrix for the graph of non-stop flights between the indicated cities. List the cities alphabetically for the rows and columns. *(Lesson 11-2)*

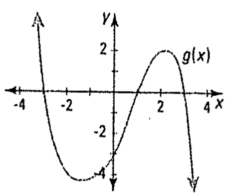


22. Suppose that the assembling of a computer consists of the following tasks.

Task		Time required (hours)	Prerequisite tasks
A	Assemble memory & CPU chips	2	
B	Assemble I/O port components	3	
C	Assemble computer circuit board	3	*A, B*
D	Assemble disk drive	3	
E	Assemble computer	2	*C, D*
F	Assemble picture tube	5	
G	Assemble monitor circuit board	3	
H	Assemble monitor	4	*F, G*
I	Assemble key mechanism	3	
J	Assemble keyboard circuit board	2	
K	Assemble keyboard	1	*I, J*
L	Package computer and peripherals	1	*E, H, K*

Draw a digraph to determine the minimum time required for the entire assembling process. *(Lesson 11-1)*

23. Solve over the complex numbers: $20x^4 + 11x^2 - 3 = 0$. *(Lesson 3-6)*



24. The graph of a function g is shown at the left. Approximate the values of x satisfying each condition. *(Lessons 2-1, 2-3)*

a. $g(x) > 0$

b. $g(x) < 0$

c. g is increasing

25. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, calculate $A \cdot A$ (which is A^2). *(Previous course)*

Exploration

26. Consider the statement: *If G is a simple graph with m edges and n vertices, then* $m \le \frac{n(n-1)}{2}$.

a. Write its contrapositive.

b. Prove or disprove either the statement or its contrapositive.

Introducing Lesson 11-4

Euler Circuits

IN-CLASS ACTIVITY

Here are definitions of four terms.

Suppose that G is a graph and v and w are vertices of G.

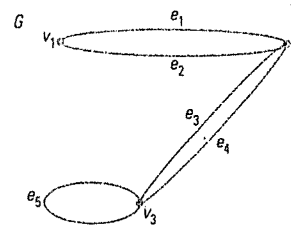
A **walk from v to w** is an alternating sequence of adjacent vertices and edges of G beginning with v and ending with w.

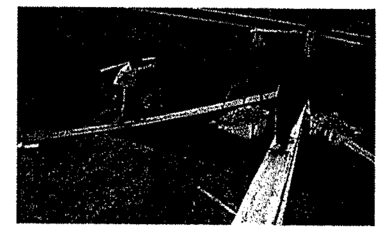
A **path from v to w** is a walk in G from v to w in which no edge is repeated.

A **circuit** is a path in G that starts and ends at the same vertex.

An **Euler circuit** is a circuit that contains every edge and vertex of G.

1 Consider the graph G shown here.





The following is a walk from v_3 to v_1: $v_3\, e_5\, v_3\, e_3\, v_2\, e_4\, v_3\, e_3\, v_2\, e_1\, v_1$. Is it a path? Why or why not?

2 When there is no confusion, we list only the edges of a walk. For the walk of Task 1, we can write $e_5\, e_3\, e_4\, e_3\, e_1$. Name three paths from v_3 to v_1.

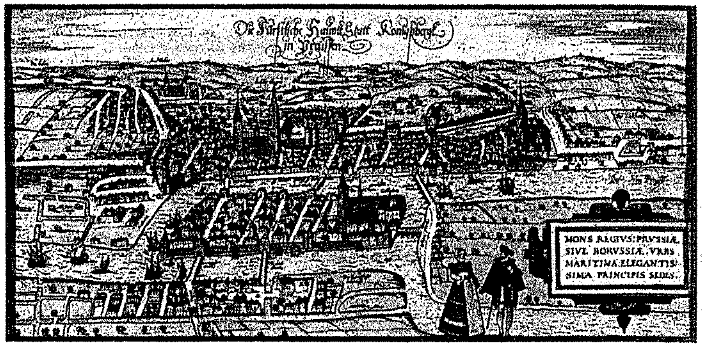
3 **a.** Name two circuits from v_1 to v_1.
b. Name an Euler circuit in G, starting at v_1.

4 Copy the table below. Fill in each cell of the table with one of the words "always," "sometimes," or "never."

	repeated edge?	starts and ends at the same point?	includes every edge and vertex?
walk			
path			
circuit			
Euler circuit			

LESSON

11-4 The Königsberg Bridge Problem



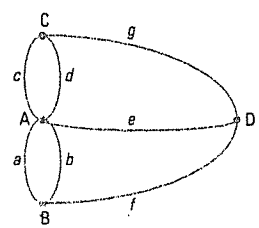
This historic etching, picturing Königsberg (now Kaliningrad), predates Euler. Note that only six bridges existed; the seventh was built later.

Return to the Problem

At the beginning of this chapter, the Königsberg bridge problem was posed:

> In the city of Königsberg, is it possible for a person to walk around the city traversing each bridge exactly once, starting and ending at the same point?

It was pointed out in Lesson 11-1 that this problem is equivalent to the question of whether it is possible to trace the graph below without picking up your pencil, traversing each edge exactly once and starting and ending at the same vertex. Using the terminology in the In-class Activity, the Königsberg bridge problem asks if there is an Euler circuit for this graph.



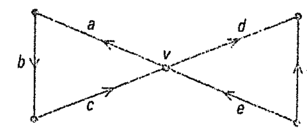
Euler's Solution

Euler proved the following theorem. It gives a necessary condition for a graph to have an Euler circuit, and it solves the Königsberg bridge problem.

Euler Circuit Theorem
If a graph has an Euler circuit, then every vertex of the graph has even degree.

Proof
Suppose G is any graph that has an Euler circuit. To show that every vertex of G has even degree, we take any particular but arbitrarily chosen vertex v and show that the degree of v must be even.

Either v is or is not the vertex where the Euler circuit starts and stops. If v is not, then each time the circuit enters v on one edge, it must leave v by a different edge. Thus the edges having v as an endpoint must occur in entry/exit pairs. This idea is illustrated by the diagram below.



In an Euler circuit, each time v is entered by one edge it is exited by another edge.

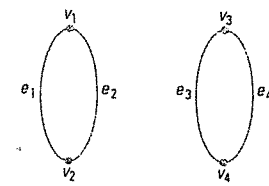
If v is the initial vertex of the Euler circuit, then the first edge of the circuit (that leads out of v) can be paired with the last edge of the circuit (that leads into v). Also, as above, any other edges having v as an endpoint occur in entry/exit pairs.

It follows that, regardless of whether or not the circuit starts and ends at v, all the edges having v as an endpoint can be divided into pairs. Therefore, the number of such edges is even, which is what was to be shown.

Now examine the graph of the Königsberg bridges. All four vertices have odd degree. Since just one vertex of odd degree is enough to preclude there being an Euler circuit, the answer to the question of the Königsberg bridge problem is "No." The Königsberg bridge graph has no Euler circuit.

Connected Graphs

Does the converse of the Euler Circuit Theorem hold true? Is it true that if every vertex of a graph has even degree, then the graph has an Euler circuit? The answer is no. The following counterexample shows a graph with four vertices v_1, v_2, v_3, and v_4 in which every vertex has even degree yet there is no Euler circuit.



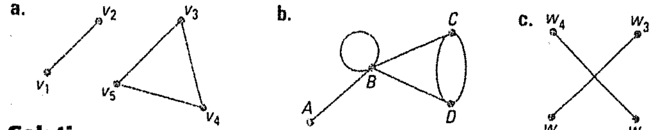
However, the converse to the Euler Circuit Theorem is true under certain conditions. To discuss these conditions requires the concept of connectedness. Roughly speaking, a graph is *connected* if it is possible to travel from any vertex to any other vertex along adjacent edges.

Definitions
Suppose G is a graph. Two vertices v and w in G are **connected** if and only if there is a walk in G from v to w. G is a **connected graph** if and only if $\forall$ vertices v and w in G, $\exists$ a walk from v to w.

Example 1

Tell whether or not the graph is connected.

a. **b.** **c.**

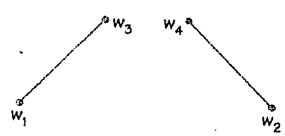


Solution

a. *The graph is not connected. It is impossible to find a walk from v_1 to v_3, for instance.*

b. *The graph is connected; each vertex is connected to each other vertex by a walk.*

c. *There is no walk from w_1 to w_2. So the graph is not connected. This can also be shown by redrawing the graph without crossings.*



When Does a Graph Have an Euler Circuit?

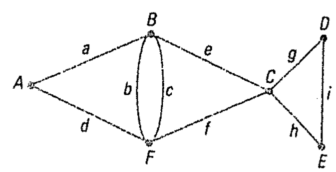
With the idea of connectedness, a sufficient condition for an Euler circuit can be given.

Theorem (Sufficient Condition for an Euler Circuit)
If a graph *G* is connected and every vertex of *G* has even degree, then *G* has an Euler circuit.

The proof is omitted because it is quite long.

Example 2

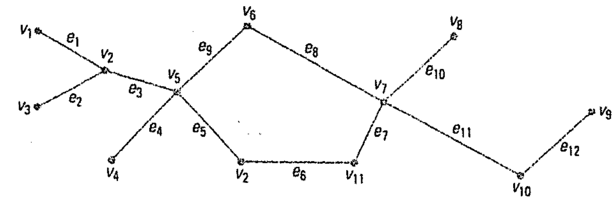
Does the following graph have an Euler circuit? If so, find such a circuit.



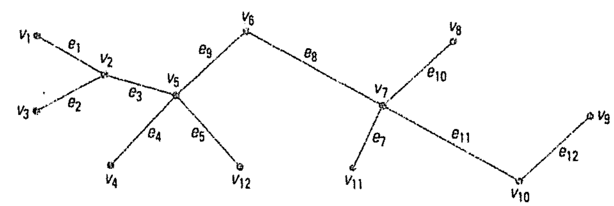
Solution

This graph is connected and every vertex has even degree:
deg(A) = deg(D) = deg(E) = 2, deg(B) = deg(C) = deg(F) = 4.
Hence this graph has an Euler circuit. One such circuit starting at A is a b c e g i h f d.

Now consider a connected graph which contains a circuit.



This graph has the following circuit starting at v_6: $e_9e_5e_6e_7e_8$. Suppose one edge is removed from the circuit. For instance, suppose edge e_6 is removed. Is the resulting graph connected?



The answer, of course, is yes. Although the direct connection from v_{12} to v_{11} has been removed, the indirect connection obtained by traveling the other way around the circuit $e_5\ e_9\ e_8\ e_7$ remains. By the same reasoning, any other pair of vertices connected using the original edge e_6 can also be connected by using this indirect route.

This discussion can be formalized to prove the following theorem, which seems obvious but has a nonobvious application in Lesson 11-7.

Theorem (Circuits and Connectedness)
If a connected graph contains a circuit and an edge is removed from the circuit, then the resulting graph is also connected.

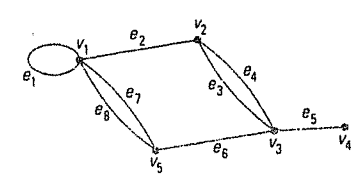
QUESTIONS

Covering the Reading

In 1 and 2, refer to the graph pictured below. A walk is given. **a.** Is it a path? **b.** Is it a circuit? **c.** Is it an Euler circuit?

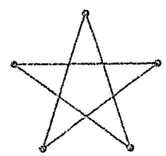
1. $v_5\ e_7\ v_1\ e_8\ v_5\ e_7\ v_1$

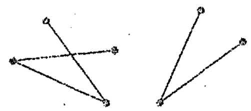
2. $e_4\ e_6\ e_7\ e_2$

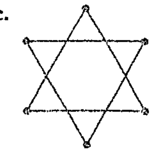


3. a. Write the contrapositive of the Euler Circuit Theorem.
   b. How is the contrapositive used to test whether a given graph has an Euler circuit?

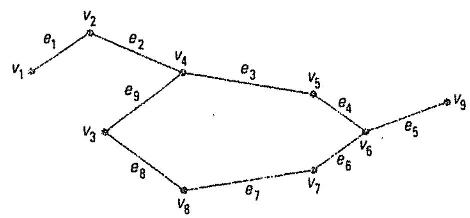
4. Tell whether the graph pictured is connected.

a. 

b. 

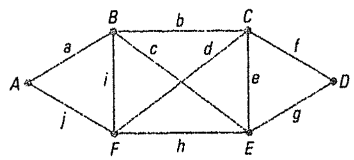
c. 

5. In the graph pictured below, the walk $e_1\,e_2\,e_3\,e_4\,e_5$ connects v_1 and v_9.



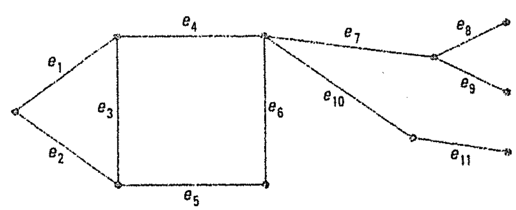
Now consider the graph obtained by removing edge e_3. Find a walk in this graph that connects v_1 and v_9.

6. Does the graph at the right have an Euler circuit? If so, describe the circuit.

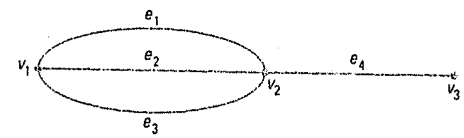


Applying the Mathematics

7. a. In the graph pictured below, list each edge that could be removed individually without disconnecting the graph.
   b. What is the maximum number of edges that can be removed at the same time without disconnecting the graph?



8. Consider the graph pictured below.
   a. Find all paths from v_1 to v_3.
   b. Find five walks from v_1 to v_3 that are not paths.
   c. How many walks from v_1 to v_3 have no repeated vertex?
   d. Can you list all possible walks from v_1 to v_3? Explain.

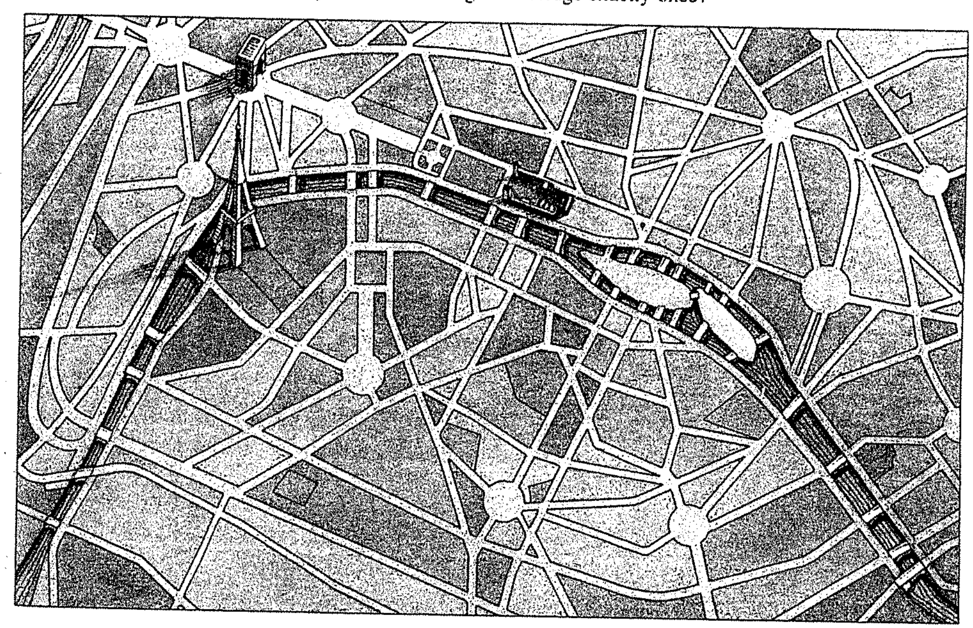


9. If a graph has an Euler circuit, must the graph be connected? Explain your answer.

10. Suppose G is a graph with five vertices of degrees 2, 2, 2, 4, and 4. Answer *yes*, *no*, or *not necessarily* to the question: Does G have an Euler circuit? Justify your answer.

11. Could a citizen of Königsberg have taken a walk around the city crossing each bridge exactly twice before returning to the starting point? Explain your answer.

12. Paris, France, is built along the banks of the Seine river and includes two islands in the river. The map below shows the bridges of Paris. Is it possible to take a walk around Paris starting and ending at the same point and crossing each bridge exactly once?



13. If a graph contains a walk from one vertex v to a different vertex w, must it contain a path from v to w? Explain your answer.

Review

14. Show, with a graph, that it is possible for 5 people on a committee each to shake the hands of two others. *(Lesson 11-3)*

15. Write the adjacency matrix for the vertices of a tetrahedron. *(Lesson 11-2)*

16. Assume that there is a test for cancer which is 98 percent accurate; that is, if someone has cancer, the test will be positive (signifying a cancer) 98 percent of the time, and if one doesn't have it, the test will be negative (signifying no cancer) 98 percent of the time. Also assume that 0.5% of the population has cancer.

a. Draw a probability tree and label its edges with probabilities to represent this situation.

b. Imagine that you are tested for cancer. What is the probability that the test will be positive?

c. What is the probability that if the test is positive, then you have cancer?

d. What is the probability that if the test is positive, then you don't have cancer? *(Lesson 11-1)*

17. Who founded the subject of graph theory and in what century did he live? *(Lesson 11-1)*

18. How many whole numbers less than 10,000 have the property that the sum of their digits is 7? *(Lesson 10-7)*

19. Suppose $\frac{\pi}{2} < x < \pi$ and $\csc x = 5$. Find $\tan x$. *(Lesson 5-7)*

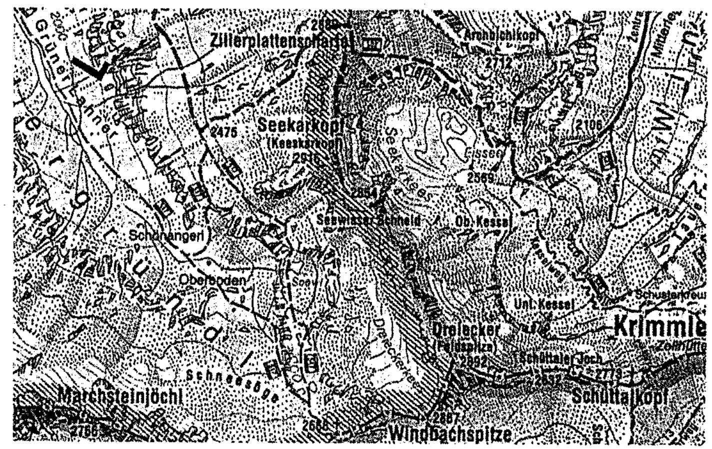
20. Describe a transformation that transforms the graph of $y = e^x$ onto the graph of $y = 3e^{x-5} + 4$. *(Lesson 3-8)*

Exploration

21. Let K_n denote the complete graph with n vertices. For what values of n does K_n have an Euler circuit? Justify your answer.

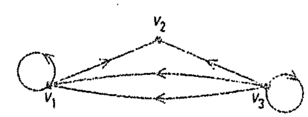
LESSON

11-5 Matrix Powers and Walks



How many paths? *This map of part of the Austrian Alps shows trails (in red) for hikers. Matrices can be used to represent the choices of paths.*

In a matrix A, the element in the ith row and jth column is often denoted by a_{ij}. This notation is quite useful with adjacency matrices. For example, consider the directed graph pictured below along with its adjacency matrix.

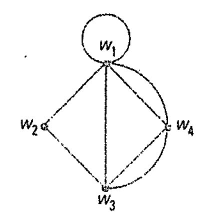


$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix} \end{array} = A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The Length of a Walk

The **length** of a walk is defined to be the number of edges in the walk. Then the entry $a_{31} = 2$ can be interpreted as indicating that there are two walks of length 1 from v_3 to v_1. The first number of the subscript in a_{31} indicates the starting vertex; the second number, the ending vertex. The entry $a_{33} = 1$ indicates that there is one walk of length 1 from v_3 to itself. The entries a_{ii} make up the **main diagonal** of the matrix, the diagonal from upper left to lower right.

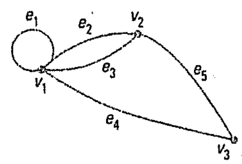
Now consider the undirected graph and its adjacency matrix B pictured below.



$$B = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$

main diagonal

Notice that this matrix has an interesting characteristic. The entries are symmetric to the main diagonal. For instance, $b_{12} = b_{21}$ and $b_{32} = b_{23}$. This is true since an edge from w_i to w_j also goes from w_j to w_i. Every matrix representation of an undirected graph has this characteristic. Such a matrix is called **symmetric**.



Example 1

How many walks of length 2 are there from v_1 to v_3 in the graph at the left?

Solution

A walk of length 2 from v_1 to v_3 will go through an "intermediate" vertex. *There are two such walks with v_2 as the intermediate vertex (e_2e_5 and e_3e_5). There is one such walk with v_1 as the intermediate vertex (e_1e_4), but none with v_3 as the intermediate vertex. Thus, there are three walks of length 2 from v_1 to v_3.*

Walks of Length 2

The length of a walk has a wonderful connection with matrices. Consider the adjacency matrix A of the graph in Example 1.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

Each entry a_{ij} gives the number of walks of length 1 from vertex v_i to vertex v_j. Thus $a_{21} = 2$ indicates that there are 2 walks of length 1 from vertex v_2 to vertex v_1. Now multiply A by itself.

$$A \cdot A = A^2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The element a_{13} in A^2 is the product of row 1 and column 3, by the rule for matrix multiplication.

$$a_{13} = [1 \quad 2 \quad 1]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \cdot 1 + 2 \cdot 1 + 1 \cdot 0 = 3$$

Notice that this computation of a_{13} also computes the number of walks of length 2 from v_1 to v_3:

$$\begin{bmatrix} \text{number of walks of length 1} \\ \text{from } v_1 \text{ to } v_1 \end{bmatrix} \times \begin{bmatrix} \text{number of walks of length 1} \\ \text{from } v_1 \text{ to } v_3 \end{bmatrix}$$
$$+ \begin{bmatrix} \text{number of walks of length 1} \\ \text{from } v_1 \text{ to } v_2 \end{bmatrix} \times \begin{bmatrix} \text{number of walks of length 1} \\ \text{from } v_2 \text{ to } v_3 \end{bmatrix}$$
$$+ \begin{bmatrix} \text{number of walks of length 1} \\ \text{from } v_1 \text{ to } v_3 \end{bmatrix} \times \begin{bmatrix} \text{number of walks of length 1} \\ \text{from } v_3 \text{ to } v_3 \end{bmatrix}$$

The entire matrix A^2 is computed in the same way.

$$A^2 = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 5 & 2 \\ 3 & 2 & 2 \end{bmatrix}$$

The entry in row 1 and column 3 of A^2 is 3, the number of walks of length 2 between v_1 and v_3 that was found in Example 1. Using similar reasoning, since $a_{22} = 5$, there are 5 walks of length 2 from v_2 to v_2. You should try to find these 5 walks.

Activity

Describe the 5 walks of length 2 from v_2 to v_2.

Walks of Length n

The discussion on page 688 is a special case of the following wonderful theorem, whose proof is too long to be included. (It can be proved using mathematical induction.) As usual,

$$A^n = \underbrace{A \cdot A \cdot \ldots \cdot A.}_{n \text{ factors}}$$

Theorem

Let G be a graph with vertices $v_1, v_2, \ldots, v_m$, and let n be a positive integer. Let A be the adjacency matrix for G. Then the element a_{ij} in A^n is the number of walks of length n from v_i to v_j.

Example 2

Determine the number of walks of length 3 between v_1 and v_2 in the graph of Example 1.

Solution

The answer is given by the element a_{12} of A^3.

$$A^3 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 15 & 15 & 9 \\ 15 & 8 & 8 \\ 9 & 8 & 5 \end{bmatrix} \text{ and so } a_{12} = 15.$$

Thus, there are 15 walks of length 3 between v_1 and v_2.

Check

The walks can be listed. Here are 12 of them. Which three are missing?

$e_1e_1e_2$	$e_1e_1e_3$	$e_2e_3e_2$	$e_3e_2e_3$	$e_2e_2e_2$	$e_3e_3e_3$	$e_1e_4e_5$
$e_2e_5e_5$	$e_3e_5e_5$	$e_4e_4e_3$	$e_4e_4e_2$	$e_2e_2e_3$		

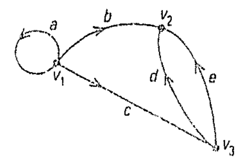
It can be proved that the powers of a symmetric matrix are symmetric. Since the matrix A of Example 1 is symmetric, its cube A^3 in Example 2 should also be symmetric. This provides another way of checking the multiplication in Example 2.

QUESTIONS

Covering the Reading

1. *True or false.* If in a graph there are 2 walks from v_2 to itself, then the entry in the second row, second column of the adjacency matrix for the graph will be 4.

2. In the adjacency matrix of the graph at the right, the entry a_{32} = __?__ and a_{23} = __?__.



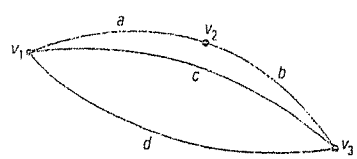
In 3 and 4, consider the matrix B shown at the right.

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \end{bmatrix} \end{array}$$

3. Give the element.
   a. b_{21} = __?__
   b. b_{13} = __?__
   c. b_{33} = __?__

4. *True or false.* The matrix represents an undirected graph.

5. *True or false.* There is only one walk from v_1 to v_2 in the graph at the right.

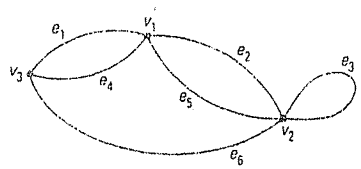


6. Describe the 5 walks of length 2 from v_2 to v_2 in the graph of Example 1.

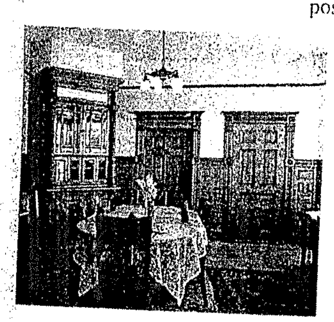
7. Describe all of the walks of length 3 from v_3 to v_3 in Example 2.

8. In Example 2, find the 3 missing walks of length 3 from v_1 to v_2.

9. Find the number of walks of length 2 from v_3 to v_2 in the graph at the right.



10. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 0 \end{bmatrix}$, calculate A^2 and A^3.

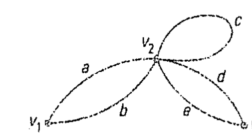


itting room in the 1875 home f Senator Wilbur Sanders in lelena, Montana

Applying the Mathematics

11. If all the entries on the main diagonal of the adjacency matrix for a graph are zero and the other entries are zero or one, then the graph is a simple graph. Explain why.

12. Determine the total number of walks of length 3 for the graph given at the right.



13. Refer to the graph on page 680. Determine the number of walks over exactly 3 bridges of Königsberg that begin and end at C.

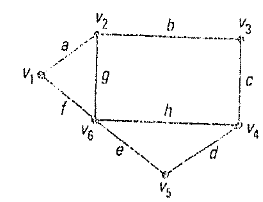
14. Consider the following true statement:

    If a graph is not directed, then its adjacency matrix is symmetric.

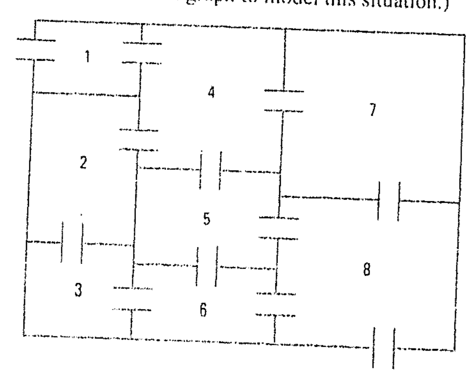
    a. Write the converse.
    b. Give a counterexample to show that the converse is false.

Review

15. Does the graph at the right have an Euler circuit? If so, find it. If not, draw an edge that will make an Euler circuit possible. *(Lesson 11-4)*



16. A house is open for public viewing. An outline of the floor plan is shown below. Is it possible to enter into room 1, pass through every interior doorway of the house exactly once, and exit from room 8? If so, how can this be done? If not, where could you put a new door to make such a tour possible? (Hint: Construct a graph to model this situation.) *(Lesson 11-4)*



17. If there are 27 people at a party, is it possible for each one to shake hands with exactly 4 other people? *(Lesson 11-3)*

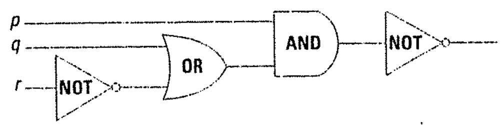
18. Draw a graph which has the adjacency matrix at the right. *(Lesson 11-2)*

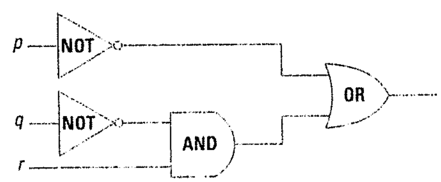
$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

19. Suppose that $2 - i$ is the fourth root of some complex number z. Find and graph the other fourth roots. *(Lessons 8-6, 8-7)*

20. Simplify the expression $\left(\frac{1 + z^{-2}}{1 - z^{-2}}\right) \cdot (1 + z)$, and state the restrictions on z. *(Lesson 5-1)*

21. Show that the two computer logic networks given below are equivalent. *(Lessons 1-3, 1-4)*





22. Solve the following systems. *(Previous course)*

a. $\begin{cases} 3x + 4y = 10 \\ 2x - y = 1 \end{cases}$

b. $\begin{cases} 8x = 18y - 12 \\ 24 = -16x + 36y \end{cases}$

Exploration

23. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

a. Calculate A^n for $n = 1, 2, 3, 4, \ldots$. Find a pattern in A^n.

b. What does this pattern tell you about the directed graph whose adjacency matrix is A?

c. Draw the graph to confirm your answer to part **b**.

LESSON

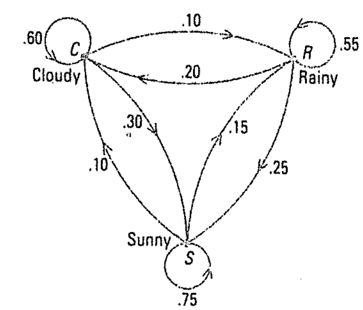
11-6 Markov Chains

Our Milky Way Galaxy. *Some astronomers have used Markov chains to study galaxies and their distribution in the universe. See page 696.*

In this lesson, graphs and powers of matrices are combined with probability, limits, and systems of equations in a display of the interconnectedness of mathematics. The ideas of this lesson have wide applicability.

A Markov Chain Weather Situation

Suppose that weather forecasters in a particular town have come up with data, represented in the following directed graph, concerning the probabilities of occurrence of sunny days (S), rainy days (R), and cloudy days (C).



Interpret this directed graph as follows. The loop about point C, labeled .60, means that 60% of the time a cloudy day is followed by another cloudy day. The .10 by edge (C, R) means that 10% of cloudy days are followed by a rainy day. The .30 by edge (C, S) means that 30% of cloudy days are followed by a sunny day.

Now suppose today is cloudy. What will the weather be two days from now?

To answer this question, represent the graph by a matrix T, where the rows and columns of T are labeled C, S, and R. The entries of T are probabilities that one type of weather on one day is followed by a particular type of weather the next day. For instance, $t_{23} = .15$ because t_{23} is in row S and column R and 15% of sunny days are followed by a rainy day.

$$\begin{array}{cc} & \begin{array}{ccc} C & S & R \end{array} \\ \begin{array}{c} C \\ S \\ R \end{array} & \begin{bmatrix} .60 & .30 & .10 \\ .10 & .75 & .15 \\ .20 & .25 & .55 \end{bmatrix} \end{array} = T$$

Notice that each element is nonnegative (since each is a probability), and that the entries in each row add to 1 (since the next day is always either cloudy, sunny, or rainy). A matrix with these properties is called a **stochastic matrix.** We call this matrix T to indicate that it contains the **transition probabilities** from one time period to the next.

In Lesson 11-5, you saw that the square of the adjacency matrix for a graph represents the number of walks of length two from one vertex to another. Here the square of T has a similar interpretation: its elements are the probabilities connecting weather two days apart.

$$T^2 = T \cdot T = \begin{bmatrix} .60 & .30 & .10 \\ .10 & .75 & .15 \\ .20 & .25 & .55 \end{bmatrix} \cdot \begin{bmatrix} .60 & .30 & .10 \\ .10 & .75 & .15 \\ .20 & .25 & .55 \end{bmatrix} = \begin{bmatrix} .410 & .430 & .160 \\ .165 & .630 & .205 \\ .255 & .385 & .360 \end{bmatrix}$$

Notice that the entries in each row still add up to 1, so T^2 is also a stochastic matrix. Reading across the first row of T^2 shows that if today is cloudy, there is a 41% chance that it will be cloudy two days from now, a 43% chance that it will be sunny, and a 16% chance of rain.

T^2 can be multiplied by itself to yield T^4, which indicates the probabilities of various types of weather occuring 4 days later. Similarly $T^4 \cdot T^4 = T^8$ and $T^8 \cdot T^2 = T^{10}$. In general, each entry of T^k indicates the probability that one type of weather will be followed by a particular type k days later.

$$T^4 \approx \begin{bmatrix} .27985 & .50880 & .21135 \\ .22388 & .54678 & .22935 \\ .25988 & .49080 & .24932 \end{bmatrix}$$

$$T^8 \approx \begin{bmatrix} .24715 & .52432 & .22853 \\ .24466 & .52544 & .22990 \\ .24740 & .52295 & .22965 \end{bmatrix}$$

$$T^{10} \approx \begin{bmatrix} .24612 & .52458 & .22930 \\ .24563 & .52474 & .22962 \\ .24628 & .52426 & .22946 \end{bmatrix} \approx \begin{bmatrix} .25 & .52 & .23 \\ .25 & .52 & .23 \\ .25 & .52 & .23 \end{bmatrix}$$

The three rows of T^{10} are almost identical. This means that no matter what the weather is today, there is approximately a 25% chance of a cloudy day 10 days from now, a 52% chance of a sunny day, and a 23% chance of rain.

Weather is dependent on many factors. The key assumption in the model used here is that the probability of a certain type of weather tomorrow is only dependent on the weather today. When a situation can exist in only a finite number of states (above there are 3 states: C, S, and R), and the probabilities of having one state precede another depend only on the earlier state, then the situation is said to be an example of a **Markov chain.**

Markov chains are named after the Russian mathematician who first studied them, Andrei Andreevich Markov (1856-1922). Markov worked in a variety of areas of mathematics, with his greatest contributions being in the area of probability theory. He developed the concept of Markov chain from the theory of probability and applied it to a study of the distributions of vowels and consonants in Russian literature. His work is frequently considered to be the first research in mathematical linguistics, the mathematical study of language structure.

Squares of Stochastic Matrices Are Stochastic

Recall that for the stochastic matrix T on the previous page, T^2 is also stochastic. In general, the kth power of any stochastic matrix is stochastic. This can be seen for the 2nd power of a 2×2 stochastic matrix as follows.

Because the entries in each row add to 1, the matrix has the form $\begin{bmatrix} x & 1-x \\ y & 1-y \end{bmatrix}$, where $0 \le x \le 1$ and $0 \le y \le 1$. Its square is

$$\begin{bmatrix} x & 1-x \\ y & 1-y \end{bmatrix} \cdot \begin{bmatrix} x & 1-x \\ y & 1-y \end{bmatrix} = \begin{bmatrix} x^2 + y - xy & 1 - x^2 - y + xy \\ xy + y - y^2 & 1 + y^2 - y - xy \end{bmatrix},$$

which is also stochastic.

Furthermore, if a stochastic matrix T has no zero entries, the rows of T^k will be nearly identical for large k. This indicates that over the long term the proportions of the occurrences of the different states stabilizes. You also saw this for the matrix T on the previous page. The rows of T^{10} are nearly identical.

Theorem (Convergence of Powers)
Let T be an $n \times n$ stochastic matrix with no zero entries. Then $\lim_{k \to \infty} T^k$ is a stochastic matrix with n identical rows.

A Markov Chain Situation in Biology

Stable populations occur in populations of plants and animals, as the example on page 696 illustrates.

Example

Consider a variety of rose that can have either a pale hue or a brilliant hue. It is known that seeds from a pale blossom yield plants of which 60% have pale flowers and 40% have brilliant flowers. Seeds from a brilliant flower yield plants of which 30% are pale and 70% are brilliant. After several generations of plants, what will be the proportion of pale and brilliant flowering plants?

Solution

The transition matrix for this situation is

$$\text{FLOWER}\quad \begin{array}{r} \\ \text{Pale} \\ \text{Brilliant} \end{array} \begin{array}{c} \text{OFFSPRING} \\ \begin{array}{cc} \text{Pale} & \text{Brilliant} \end{array} \\ \begin{bmatrix} .6 & .4 \\ .3 & .7 \end{bmatrix} \end{array} = T.$$

Let a and b be the proportion of plants with pale and brilliant flowers, respectively, when the population stabilizes. Then $a + b = 1$, since no other flowers are possible. Yet the proportion of the flowers of the next generation that are pale will be $.6a + .3b$ because .6 of those produced by the pale flowers are pale, and .3 of those produced by the brilliant ones are pale. But since the population has stabilized, the fraction of the next generation that is pale must still be a. This results in the equation

$$.6a + .3b = a.$$

Thus the following system must be satisfied.

$$\begin{cases} .6a + .3b = a \\ a + b = 1 \end{cases}$$

This system has the solution $(a, b) = \left(\frac{3}{7}, \frac{4}{7}\right) \approx (.43, .57)$. *So when the population stabilizes, about 43% of the plants will have pale flowers and 57% will have brilliant flowers.*

Check 1

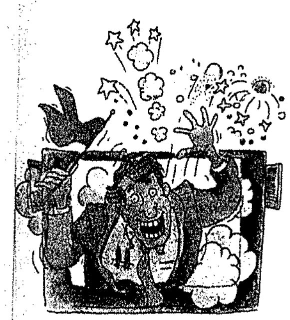
The fraction of flowers in the next generation that are brilliant is $.4a + .7b$. When the population has stabilized, the fraction that is brilliant is b. So we must have $.4a + .7b = b$. This is true when $a = \frac{3}{7}$ and $b = \frac{4}{7}$.

Check 2

The result matches the result obtained by calculating powers of T. For instance,

$$T^{10} \approx \begin{bmatrix} .42857 & .57143 \\ .42857 & .57143 \end{bmatrix} \approx \begin{bmatrix} \frac{3}{7} & \frac{4}{7} \\ \frac{3}{7} & \frac{4}{7} \end{bmatrix}.$$

After Markov published his theory, his techniques were adopted by scientists in a wide range of fields. Albert Einstein used these ideas to study the Brownian motion of molecules. Physicists have employed them in the theory of radioactive transformation, nuclear fission detectors, and the theory of tracks in nuclear emulsions. Astronomers have used Markov theory to study fluctuations in the brightness of the Milky Way and the spatial distribution of galaxies. Biologists have used Markov chains to describe population growth, evolution, molecular genetics, pharmacology, tumor growth, and epidemics. Sociologists have modeled voting behavior, geographical mobility, growth and decline of towns, sizes of businesses, changes in personal attitudes, and deliberations of trial juries with Markov chains.



QUESTIONS

Covering the Reading

In 1–4, consider the weather situation on page 693.

1. a. If it is sunny today, what is the probability that tomorrow is rainy?
   b. If it is sunny today, what is the probability that tomorrow is sunny?
2. a. If it is rainy today, what is the probability that two days from now is rainy?
   b. If it is sunny today, what is the probability that four days from now is rainy?
3. Is T^{10} a stochastic matrix?
4. a. In the matrix T^{10}, what does the number .24612 represent?
   b. What is the significance of the fact that the rows of T^{10} are nearly identical?

In 5–8, consider the flower situation of this lesson's Example.

5. a. If a rose is brilliant, what is the probability that its offspring are brilliant?
   b. If a rose is pale, what is the probability that its offspring are pale?
6. Solve the system to verify the solution.
7. Verify Check 1.
8. Using $T^{10} = \begin{bmatrix} \frac{3}{7} & \frac{4}{7} \\ \frac{3}{7} & \frac{4}{7} \end{bmatrix}$, calculate T^{20} and explain your result.

Applying the Mathematics

9. At each four-month interval, two TV stations in a small town go through "ratings week." They try to offer special programs which will draw viewers from the other station. During each period, MBC (Markov Broadcasting Company) wins over 20% of SBS (Stochastic Broadcasting System) viewers, but loses 10% of its viewers to SBS.
   a. Draw a graph (like that shown at the beginning of this lesson) to represent the movement of viewers between stations.
   b. Write down the transition matrix.
   c. Using the method of the rose example, find the long-term distribution of viewers watching each station.

10. The British scientist Sir Francis Galton studied inheritance by looking at distributions of the heights of parents and children. In 1886 he published data from a large sample of parents and their adult children showing the relation between their heights. The following matrix is based on his data. Since he had to use volunteers in his study, he could not be sure that his sample accurately reflected the English population.

$$\begin{array}{ll} & \text{CHILD} \\ & \begin{array}{ccc}\text{Tall} & \text{Med} & \text{Short}\end{array} \\ \text{PARENT}\ \begin{array}{r}\text{Tall}\\ \text{Med}\\ \text{Short}\end{array} & \begin{bmatrix} .53 & .32 & .15 \\ .30 & .34 & .36 \\ .15 & .32 & .53 \end{bmatrix} = T \end{array}$$

According to this matrix,

$$T^2 \approx \begin{bmatrix} .399 & .326 & .274 \\ .315 & .327 & .358 \\ .255 & .326 & .419 \end{bmatrix} \text{ and } T^{10} \approx \begin{bmatrix} .321 & .327 & .353 \\ .321 & .327 & .353 \\ .321 & .327 & .353 \end{bmatrix}.$$

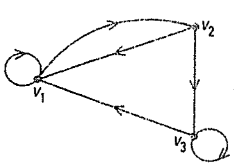
a. What proportion of the children of tall parents were short?
b. Use T^2 to tell what proportion of grandchildren of tall people were short.
c. Use T^{10} to predict the approximate proportion of tall, medium, and short people in the population in the long run.

11. Prove for 2×2 matrices: *If A is stochastic and B is stochastic, the product AB must be stochastic.*

12. Consider the matrix T at the right. $\begin{bmatrix} .5 & .5 \\ .4 & .6 \end{bmatrix}$
a. Is T stochastic?
b. Calculate T^2, T^4, T^8, and T^{16}.
c. Find two numbers a and b such that $vT = v$ where $v = [a \quad b]$ and $a + b = 1$.
d. What do a and b represent?

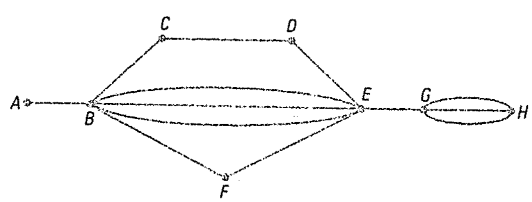
13. Generalize the result of Question 8 and prove your generalization.

Review

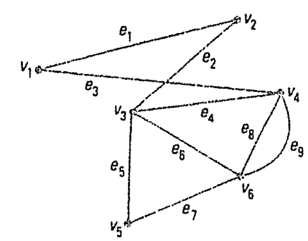


14. Find the total number of walks of length 4 which end at v_1 in the directed graph at the left. *(Lesson 11-5)*

15. In the graph pictured below, determine the number of paths from A to H that contain no circuits. *(Lesson 11-4)*



16. Consider the graph at the right.
a. Does the graph have an Euler circuit? Justify your answer.
b. What is the maximum number of edges that could be removed while keeping the graph connected?
(Lesson 11-4)



17. In a league of nine teams, is it possible for each team to play exactly seven other teams? Explain why or why not. *(Lesson 11-3)*

18. Suppose the height (in feet) of an object t seconds after it is thrown is given by $h(t) = -16t^2 + 50t + 10$. *(Lessons 9-3, 9-4, 9-5)*
a. Find the object's velocity 1 second after it is thrown.
b. When is the object's velocity the opposite of the velocity found in part **a**?
c. When does the object reach its maximum height?
d. How are the times in parts **a**, **b**, and **c** related?
e. When is the object's acceleration positive?

19. Solve the inequality $2\sin^2 x + \cos^2 x < \frac{1}{4}$ when $0 \le x \le 2\pi$. *(Lesson 6-8)*

20. Use limit notation to describe the end behavior of the function f given by $f(n) = \dfrac{2n^2 + 3n + 1}{6n^2}$. *(Lesson 5-5)*

21. Prove: *Exactly one of every four consecutive integers is divisible by 4.* (Hint: Use the Quotient-Remainder Theorem.) *(Lesson 4-2)*

Exploration

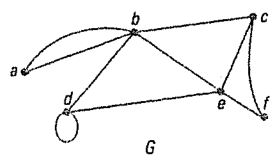
22. Find an example in a book or article that describes how a Markov chain is used in biology, linguistics, or politics.

A project presents an opportunity for you to extend your knowledge of a topic related to the material of this chapter. You should allow more time for a project than you do for a typical homework question.

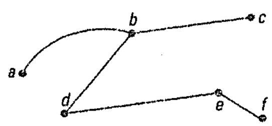
PROJECTS 11 CHAPTER ELEVEN

1 Spanning Trees

A **tree** is a connected graph that has no circuits. Given a connected graph G, a **spanning tree** is a tree consisting of a subset of the edges of G but all of the vertices of G. Thus, by definition, it is a part of G which is simple and keeps all the vertices of G connected. For example, a graph G is shown below followed by a spanning tree for G.



G

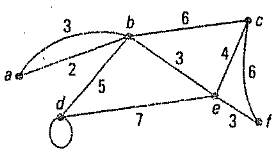


A spanning tree for G

a. Find another spanning tree for G.

b. Describe a systematic algorithm that could be used to find a spanning tree given a connected graph, and demonstrate it on a graph of your choice.

c. Suppose each edge of the graph G is labeled with a number, as is done below. Find a spanning tree for G such that the sum of the labels on its edges is the smallest possible. Such a tree is called a **minimal spanning tree**.

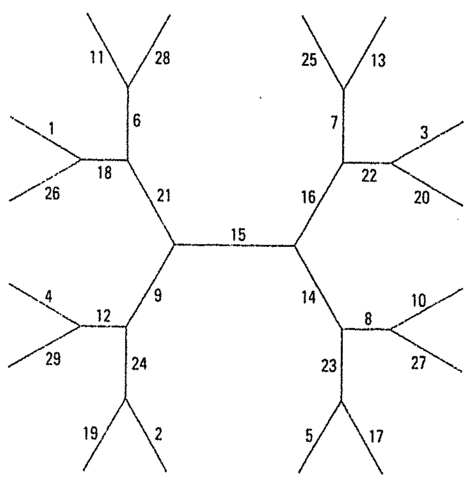


d. Find a systematic algorithm for finding a minimal spanning tree given a connected graph, and demonstrate it on a graph of your choice. (You might look up Kruskal's algorithm or Pimm's algorithm in a book.)

e. What would be some real life applications for such an algorithm?

2 Ringel's Conjecture

The following unsolved problem was posed by G. Ringel and offered by Richard K. Guy in the December 1989 issue of the *American Mathematical Monthly*. Consider the connected graph drawn below. Note that it has no circuits, and that all of its vertices have degree 1 or 3. Its 29 edges have been numbered from 1 to 29 in such a way that the sum of the numbers on the three edges leading into any vertex of degree 3 is always 45.



Ringel's conjecture is as follows:

Suppose you are given any connected graph which has no circuits and all of whose vertices have degree 1 or 3. Let n be the number of edges in the graph. Then you can

number the edges of the graph from 1 to n in such a way that the sum of the numbers on the edges leading into any vertex of degree 3 is a constant.

This conjecture has been neither proved nor disproved. Explore this problem by trying to verify the conjecture for smaller graphs. Is there a systematic way of numbering the edges? If so, can it be used to prove the conjecture? Or, can you find a counterexample?

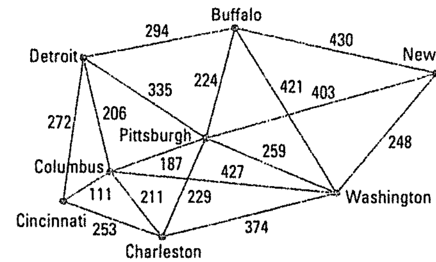
3 Constellations

Astronomers recognize 88 constellations of stars. Constellations historically were imagined patterns among the brighter stars in the nighttime sky. Each constellation can be described by a graph. Pick 50 of the 88 constellations and display their graphs.

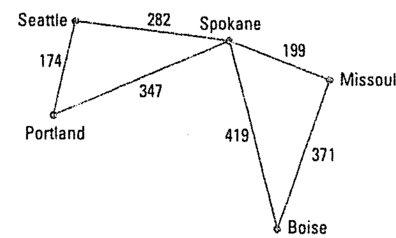
Determine whether each graph is simple or not, connected or not, traversable or not, or whether it contains circuits. Which of the graphs do you find most interesting, and why?

4 The Traveling Salesman Problem

a. Suppose a salesperson wishes to travel to each city on the map at the top of the next column exactly once, starting and ending in New York, and using only the roads shown. The numbers on the roads indicate distances (in miles) between cities. Find the shortest route that the salesperson could use.



b. Explain why it would be impossible for the salesperson to visit each city below exactly once, using only the roads shown, and end at the starting point.



c. The general problem of parts **a** and **b** is known as the *traveling salesman problem*. Refer to a book or the Internet to find out how much is known about this problem. Write a report on your findings, along with your solutions to parts **a** and **b**.

5 Dynamic Programming

Find out what *dynamic programming* is. Give some examples of problems that can be solved using dynamic programming and demonstrate their solutions. Describe the recursive algorithm that is used in solving these types of problems. For assistance, refer to books on artificial intelligence or discrete mathematics.

SUMMARY

Problems in medicine, science, and business, as well as various puzzles, can be represented and solved using graphs. Directed graphs are useful for solving scheduling problems, and probability trees are helpful in situations involving the probability that one event occurs if another event occurs.

Because the total degree of any graph is twice the number of edges, the total degree of any graph is even. Thus every graph has an even number of vertices of odd degree. These facts can be used to determine that certain types of graphs do not exist, providing solutions to a special class of problems referred to as handshake problems.

Euler proved that if a graph has an Euler circuit, then every vertex of the graph has even degree, and that if every vertex of a connected graph has even degree, then the graph has an Euler circuit. These results can be used to determine whether a given graph has an Euler circuit, and thus can be used to solve practical problems as well as puzzles such as the Königsberg bridge problem.

If an edge is removed from a circuit in a connected graph, then the graph remains connected. This theorem helps to prove Euler's formula: In any connected graph with no crossings, V vertices, E edges, and F faces, $V - E + F = 2$. This relation can be applied to any polyhedron, where V, E, and F are the number of vertices, edges, and faces of the polyhedron.

Every graph can be represented by an adjacency matrix which contains the numbers of edges from each vertex to each other vertex. The adjacency matrix for an undirected graph is always symmetric. The number of walks of length n from a given vertex to another given vertex can be obtained from the nth power of the adjacency matrix.

A Markov chain is a system involving a succession of changes from one state (or condition) to another, where the probability of moving to one state depends only on the previous state. It can be modeled by a stochastic matrix which contains those probabilities. It can be proved that for large values of n, the nth power of a stochastic matrix approaches a matrix in which every row is the same. This implies that in a Markov chain, after a long period of time, the probability of being in each state approaches a constant.

CHAPTER REVIEW

Questions on SPUR Objectives

SPUR stands for **S**kills, **P**roperties, **U**ses, and **R**epresentations. The Chapter Review questions are grouped according to the SPUR Objectives for this chapter.

SKILLS DEAL WITH THE PROCEDURES USED TO GET ANSWERS.

Objective A: *Draw graphs given sufficient information.* (*Lessons 11-2, 11-3*)

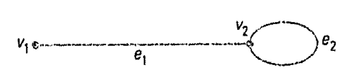
1. a. Draw a graph with three vertices and three edges.
   b. Is it possible to draw a graph with three vertices and three edges such that two edges are not adjacent to each other? If so, do it. If not, explain why not.
2. Draw a graph with two loops, an isolated vertex, and two parallel edges.
3. Draw all the simple graphs with three vertices.
4. Draw a simple graph with five vertices of the following degrees: 2, 3, 3, 4, and 4.
5. Draw the graph defined below.
   set of vertices: $\{v_1, v_2, v_3, v_4, v_5\}$
   set of edges: $\{e_1, e_2, e_3, e_4, e_5\}$
   edge-endpoint function:

edge	endpoints
e_1	$\{v_1, v_3\}$
e_2	$\{v_1, v_4\}$
e_3	$\{v_2, v_4\}$
e_4	$\{v_1\}$
e_5	$\{v_1, v_3\}$

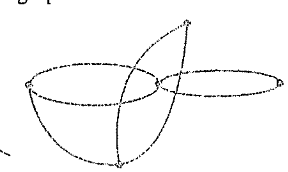
PROPERTIES DEAL WITH THE PRINCIPLES BEHIND THE MATHEMATICS.

Objective B: *Identify parts of graphs and types of graphs.* (*Lessons 11-2, 11-3, 11-4, 11-5*)

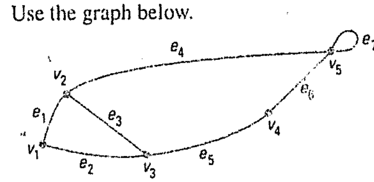
6. Use the graph below.



   a. *True or false.* There is exactly one walk from v_1 to v_2.
   b. *True or false.* There is exactly one path from v_1 to v_2.
7. Is the graph below connected?

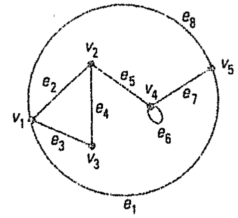


8. Use the graph below.



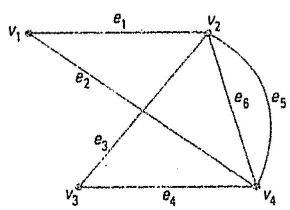
   a. Starting at v_2, consider the walk $e_4\, e_7\, e_6\, e_5\, e_3$.
      i. Is it a path?
      ii. Is it a circuit?
      iii. Is it an Euler circuit?
   b. Starting at v_2, consider the walk $e_4\, e_6\, e_5\, e_3\, e_1\, e_2\, e_3$.
      i. Is it a path?
      ii. Is it a circuit?
      iii. Is it an Euler circuit?
   c. Identify all paths from v_1 to v_3 and give their lengths.
   d. Identify three circuits that go through v_1.

9. Use the graph drawn below.



a. Identify all vertices adjacent to v_1.
b. Identify all edges adjacent to e_5.
c. Identify any isolated vertices.
d. Identify any parallel edges.
e. Identify any loops.
f. *True or false.* If edge e_6 is removed, the graph is simple.
g. Give the degree of each vertex.
h. Give the total degree of the graph.

10. Consider the graph below.



a. Identify an Euler circuit.
b. Identify two circuits that are not Euler circuits.
c. Identify a walk that is not a path.
d. What is the minimum number of edges to remove so that the graph is no longer connected?
e. What is the maximum number of edges that can be removed at the same time while keeping the graph connected? List one such set of edges.

11. Give the edge-endpoint function table for the graph in Question 9.

Objective C: *Determine whether there exists a graph containing vertices with given degrees.* *(Lesson 11-3)*

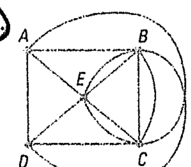
In 12–15, either draw a graph with the given properties or show that no such graph exists.

12. graph with 5 vertices of degrees 1, 2, 2, 3, and 5
13. graph with 5 vertices of degrees 1, 2, 2, 3, and 0
14. simple graph with 5 vertices of degrees 1, 2, 2, 3, and 0
15. graph with 9 vertices of degrees 0, 1, 1, 1, 2, 2, 2, 3, and 3
16. Suppose that the sum of the entries in a matrix is odd. Can this matrix be the adjacency matrix of a graph? Explain your answer.

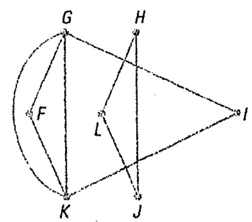
Objective D: *Determine whether a graph has an Euler circuit.* *(Lesson 11-4)*

In 17–20, determine, if possible, whether the graph has an Euler circuit. Justify your answer.

17.



18.



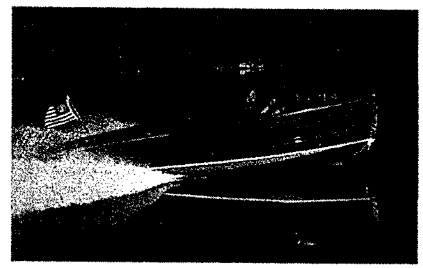
19. the graph whose adjacency matrix is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

20. a graph with vertices of degrees 2, 2, 4, and 6

USES DEAL WITH APPLICATIONS OF MATHEMATICS IN REAL SITUATIONS.

Objective E: *Use graphs to solve scheduling and probability problems.* *(Lesson 11-1)*

21. Oiler Motorboats manufactures two models of motorboat: a compact model called the Pac, and a luxury model called the Lux. In 1997, 69% of the boats sold were Oiler Pacs, and 31% were Oiler Luxes. Since then, 5% of the owners of an Oiler Pac have had to replace the rudder; of the others, 3% have had to replace the fuel gauge. The rest have needed no repairs. 7% of the owners of an Oiler Lux have had to replace the rudder; of the others, 4% the fuel gauge. The rest have needed no repairs.
   a. Draw a probability tree to represent this situation, labeling edges with the proper probabilities.
   b. If a 1997 Oiler was brought in for rudder replacement, what is the probability that it was a Pac?



22. Suppose that at any given day in a particular city, the probability that a given car is being broken into is .01%. Also suppose that a Car-Safe alarm system installed on a car sounds 96% of the time that the car is broken into, but also sounds 2% of the time that the car is not being broken into.
   a. Draw a probability tree to represent the situation.
   b. Find the probability that the car is really being broken into when the alarm sounds.

23. Suppose the process of assembling a car at a particular plant can be broken down into the following tasks.

Task		Time required (hours)	Prerequisite tasks
A	Assemble body	6	
B	Paint exterior	3	A
C	Assemble engine	11	
D	Install engine	5	B, C
E	Assemble water pump	4	
F	Assemble carburetor	5	
G	Install fuel, exhaust, electrical, cooling systems	12	D, E, F
H	Assemble interior parts	5	
I	Install interior	5	G, H

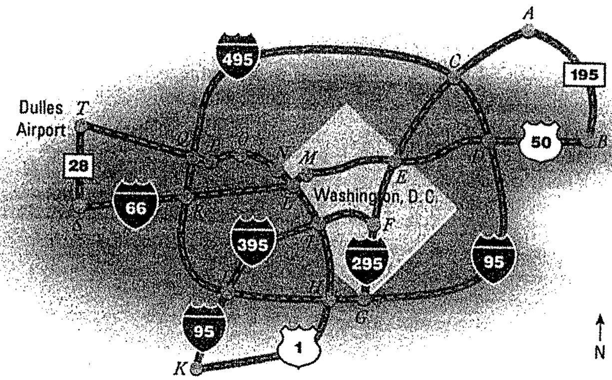
   a. Sketch a directed graph to represent the situation.
   b. What is the minimal time required to assemble a car?

Objective F: *Use the Total Degree of a Graph Theorem and its corollaries to solve handshake problems.* *(Lesson 11-3)*

24. In a class of 25 students, is it possible for each student to shake hands with exactly fifteen other students? Justify your answer.

25. From 1970 to 1975, the National Football League had two conferences each with 13 teams. If the league office had decided that every team should play 11 games in its own conference, each against a different team, would this have been possible? Justify your answer.

26. Six authors are writing a textbook, each one writing a different part. In order to maintain some unity in the book, they decide that each author should show the part he or she has written to three other authors. They want to do this in the following way: Each author will make three copies of what he or she has written, then trade each copy with a different author. Is this possible? Justify your answer.



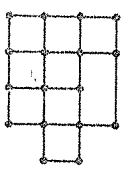
Objective G: *Solve application problems involving circuits.* (Lessons 11-1, 11-4)

27. A map of the Washington, D.C., area is shown above.

a. Explain why it is impossible to travel each road shown above exactly once and return to where you started.

b. What one section of road (that is, one edge of the graph) can be removed to make it possible?

28. Consider the map of a section of a city shown below.



a. Each corner (indicated by a dot) is a recycling pick-up point. Is there a route that a truck could follow which would begin and end at the same place and go past each of the other pick-up points exactly once? If so, find it. If not, explain why not.

b. Is there a route that a street cleaner could follow which would begin and end at the same place and travel every section of road exactly once? If so, find it. If not, explain why not.

Objective H: *Use stochastic matrices to make long-term predictions.* (Lesson 11-6)

29. Some friends like to go bowling on Tuesdays. If they go on a particular Tuesday, there is a 40% chance they will go bowling the next Tuesday. Otherwise, there is a 75% chance that they will bowl the following Tuesday.

a. Draw a directed graph representing the situation.

b. Find T, the transition matrix.

c. Estimate how often the friends bowl on average over a long period of time by calculating T^8.

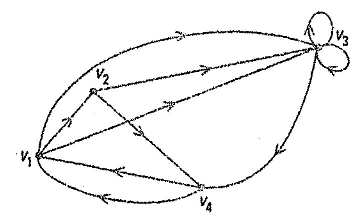
d. Find how often the friends bowl on average over a long period of time by solving a system of equations.

30. In a certain state, it was found that 60% of the daughters of women registered to vote as Democrats also register as Democrats, 15% register as Republicans, and the rest register as Independents. 70% of the daughters of Republicans are Republicans, 20% are Democrats, and the rest are Independents. 50% of the daughters of Independents are Independents, 30% are Democrats, and the rest are Republicans. Assume this pattern continues over many generations. What percentage of women will be registered in each group?

REPRESENTATIONS DEAL WITH PICTURES, GRAPHS, OR OBJECTS THAT ILLUSTRATE CONCEPTS.

Objective I: *Convert between the picture of a graph or directed graph, and its adjacency matrix.* (Lesson 11-2)

31. Write the adjacency matrix for the directed graph shown below.



32. How can you tell from its adjacency matrix whether or not a graph is simple?

33. Consider the matrix shown below.

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

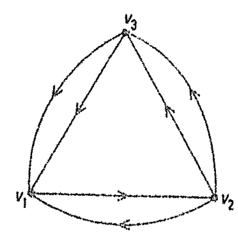
a. Draw a directed graph whose adjacency matrix is the matrix given above.

b. Could the matrix above be the adjacency matrix of a graph that is not directed? If so, draw the graph. If not, explain.

34. Draw a graph (not directed) whose adjacency matrix is given below.

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

35. In the adjacency matrix of the directed graph below, a_{13} = __?__ and a_{22} = __?__.



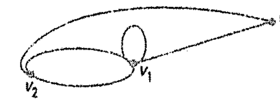
Objective J: *Use the powers of the adjacency matrix of a graph to find the number of walks of a given length, from a given starting vertex to a given ending vertex.* (Lesson 11-5)

36. The adjacency matrix for a graph is $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

How many walks of length 2 go from v_2 to v_3?

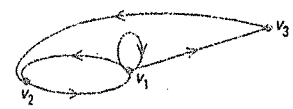
37. a. Give the adjacency matrix for the graph below.

b. How many walks of length 3 are there which start at v_1?



38. a. Give the adjacency matrix for the directed graph below.

b. How many walks of length 3 are there which start at v_1?



39. Consider the matrix $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

a. A has the property that $A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

In fact, $A^n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ for all $n \geq 4$.

What does this imply about walks in the directed graph with adjacency matrix A?

b. Confirm your answer to part **a** by drawing the directed graph with adjacency matrix A.

CHAPTER 11 REVIEW (pp. 713–716)

1. a. Sample: **See right.** b. Yes, Sample: **See right.**
3. **See right.** 5. **See right.** 7. Yes
9. a. v_2, v_3, *and* v_5
   b. e_2, e_4, e_6, e_7
   c. none
   d. e_1 and e_8
   e. e_6
   f. False
   g. $\deg(v_1) = 4$; $\deg(v_2) = 3$; $\deg(v_3) = 2$; $\deg(v_4) = 4$; $\deg(v_5) = 3$
   h. 16

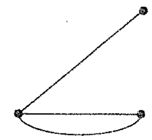
11.

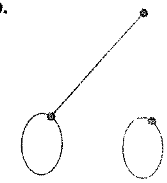
edge	endpoint
e_1	$\{v_1, v_5\}$
e_2	$\{v_1, v_2\}$
e_3	$\{v_1, v_3\}$
e_4	$\{v_2, v_3\}$
e_5	$\{v_2, v_4\}$
e_6	$\{v_4\}$
e_7	$\{v_4, v_5\}$
e_8	$\{v_1, v_5\}$

13. Sample: **See right.** 15. Impossible; a graph cannot have an odd number of odd vertices. 17. The graph has an Euler circuit by the sufficient condition for an Euler Circuit Theorem, since it is connected and every vertex is of even degree. 19. No, because v_2 and v_3 have odd degree. 21. a. **See right.** b. $\approx 61.4\%$
23. a. **See right.** b. 33 hours 25. No, a graph cannot have an odd number of odd vertices. 27. a. Vertices F and G have odd degree, so there is not an Euler circuit. b. the edge between F and G
29. a. **See right.**
   b.
$$\begin{array}{c} \\ B \\ NB \end{array}\begin{array}{c} \begin{array}{cc} B & NB \end{array} \\ \begin{bmatrix} .4 & .6 \\ .75 & .25 \end{bmatrix} \end{array}$$
   c. $T^8 \approx \begin{bmatrix} .5557 & .4443 \\ .5554 & .4446 \end{bmatrix}$
   They bowl on about 56% of the Tuesdays.
   d. $\approx 56\%$

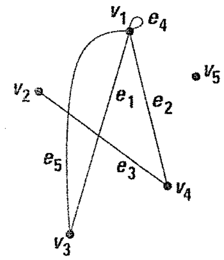
31.
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

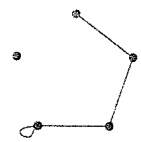
33. a. **See right.** b. No, the matrix is not symmetric.
35. a. 0 b. 0
37. a.
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array}\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$
   b. 39
39. a. There are no walks of length 4 or more. b. **See right.**

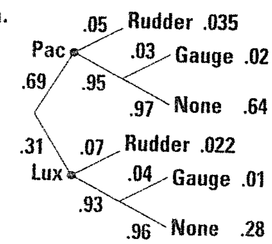
1. a.


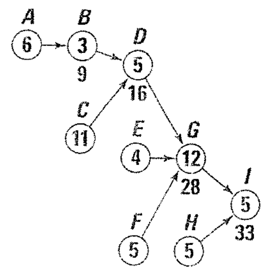
1. b.


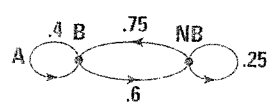
3.

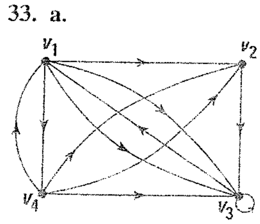

5.


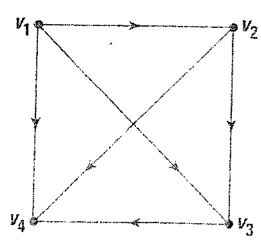
13.


21. a.


23. a.


29. a.


33. a.


39. a.


SELECTED ANSWERS